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Residue currents on analytic spaces

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Abstract

This thesis concerns residue currents on analytic spaces.

In the first paper, we construct Coleff-Herrera products and Bochner-Martinelli type currents associated with a weakly holomorphic mapping, and show that these currents satisfy well-known properties from the strongly holomorphic case. This includes the transformation law, the Poincaré-Lelong formula and the equivalence of the Coleff-Herrera product and the Bochner-Martinelli current associated with a complete intersection of weakly holomorphic functions.

In the second paper, we discuss the duality theorem on singular varieties. In the case of a complex manifold, the duality theorem, proven by Dickenstein-Sessa and Passare, says that the annihilator of the Coleff-Herrera product associated with a complete intersection f equals the ideal generated by f. We give sufficient and in many cases necessary conditions in terms of certain singularity subvarieties of the sheaf $\mathscr{O}_Z$ for when the duality theorem holds on a singular variety Z.

Keywords: analytic spaces, weakly holomorphic functions, residue currents, Coleff-Herrera products, the duality theorem

Contents

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INTRODUCTION

The theory of residue currents can be seen as a generalization of the theory of residues in one complex variable. The starting point of the theory of residue currents was the result by Herrera and Lieberman, [9], that one can define the principal value current $1/f$ and the residue current $\bar{\partial}(1/f)$ for any holomorphic function on an analytic space. This was generalized to tuples of holomorphic functions by Coleff and Herrera, [4].

We describe briefly how this works in one variable and the connection to the ordinary residue of a holomorphic function. Let f be a holomorphic function with an isolated zero at the origin in $\mathbb{C}$. One can define the principal value current $1/f$ by

$$\frac{1}{f}.\varphi = \lim_{\epsilon \to 0} \int_{|f| \geq \epsilon} \frac{\varphi}{f},$$

where φ is a test form, i.e., a smooth form with compact support. Then one defines the residue current of f, denoted $\bar{\partial}(1/f)$, as $\bar{\partial}$ of $1/f$ in the current sense, and by Stokes' theorem we have

$$\bar{\partial}\frac{1}{f}.\varphi = \lim_{\epsilon \to 0} \int_{|f| = \epsilon} \frac{\varphi}{f}. \tag{0.1}$$

If $\phi = f/g$ is a meromorphic function, the residue of ϕ at $z = 0$ is defined as

$$\operatorname{Res}_{\{z=0\}} \frac{f}{g} := \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f dz}{g},$$

where ϵ is small enough so that g is non-zero on $\{0 < |z| \leq \epsilon\}$. Then it is easy to verify that if $\chi \equiv 1$ in a neighborhood of 0 and g has an isolated zero on $\operatorname{supp} \chi$,

$$\operatorname{Res}_{\{z=0\}} \frac{f}{g} = \frac{1}{2\pi i} f \bar{\partial} \frac{1}{g}.\chi dz.$$

Residue currents have been used in various ways to study division problems (i.e., problems related to ideal membership), like effective versions of the Hilbert Nullstellensatz and the Briançon-Skoda theorem, see for example [2], [3], [12], [13] for this, and various other uses of residue currents. The general idea behind using residue currents in division problems is that one can find representation formulas which, loosely speaking, are of the form

$$\varphi(z) = \int A(z,\zeta)\varphi(\zeta) + \int B(z,\zeta) R \varphi(\zeta), \tag{0.2}$$

where $A(z,\zeta)$ lies in some ideal (f) in the z-variable, and R is a residue current associated with f, and hence, if φ annihilates R, then (0.2) gives an explicit representation of the ideal membership.

The basic example of a residue current is the Coleff-Herrera product of a tuple of holomorphic functions, as defined in [4]. It can be seen as a generalization of (0.1) to tuples of holomorphic functions.

Definition 1. Let $f=(f_1,\dots,f_p)$ be a tuple of holomorphic functions on an analytic space Z. The *Coleff-Herrera product* of f is defined by

$$\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.\varphi := \lim_{\delta\to 0}\int_{\cap\{|f_i|=\epsilon_i(\delta)\}}\frac{\varphi}{f_p\cdots f_1},$$

where φ is a test form and $\delta\mapsto\epsilon(\delta)$ is a so-called *admissible path.* Intuitively, this means that $\epsilon_1(\delta)$ tends to 0 much faster than $\epsilon_2(\delta)$ and so on, and, more precisely, that for any $k\geq 1$, there exist constants $C_{k,j}$ such that $\epsilon_j(\delta)\leq C_{k,j}\epsilon_{j+1}(\delta)^k$ for $j=1,\dots,p-1$.

The fact that $\epsilon(\delta)$ is an admissible path guarantees the existence of this limit.

An analytic subvariety $Z\subseteq\Omega\subseteq\mathbb{C}^n$ is a subset defined as the zero set of some holomorphic functions, and an analytic space is just a space which locally looks like an analytic subvariety of $\mathbb{C}^n$ (in a similar way as a complex manifold is a space which locally looks like an open set in $\mathbb{C}^n$).

Although the Coleff-Herrera product was defined on analytic spaces, most of the work on residue currents later on has been done on complex manifolds. The basic theme of this thesis is to look at classical results of residue currents and see to what extent these results extend to analytic varieties.

1. Holomorphic functions on analytic varieties

Let $Z\subseteq\Omega\subseteq\mathbb{C}^n$ be an analytic variety. On $Z_{\rm reg}$, i.e., the set of points in Z where Z is locally a complex manifold, it is clear what holomorphic functions are. However, on $Z_{\rm sing}$, i.e., the set of points where Z is not locally a complex manifold, there are different notions of holomorphicity.

The main generalization of holomorphic functions on $\mathbb{C}^n$ to an analytic variety are the strongly holomorphic functions (usually just called holomorphic functions, but here we say strongly holomorphic in contrast to weakly holomorphic functions).

Definition 2. Let $Z\subseteq\Omega\subseteq\mathbb{C}^n$ be an analytic subvariety of Ω. A function $f:Z\to\mathbb{C}$ is *strongly holomorphic*, denoted $f\in\mathcal{O}(Z)$, if it locally has a holomorphic extension to Ω.

Another generalization of holomorphic functions are the weakly holomorphic functions. These arise naturally for example in extensions of holomorphic functions across some (small) analytic set, and as functions corresponding to strongly holomorphic functions in any normal modification, for example a resolution of singularities.

Definition 3. A function $f : Z_{\text{reg}} \to \mathbb{C}$ is *weakly holomorphic*, denoted $f \in \tilde{\mathcal{O}}(Z)$, if it is holomorphic on Z_{reg}, and locally bounded at Z_{sing}.

We start with two examples of weakly holomorphic functions.

Example 1. Let $Z = \{z^3 - w^2 = 0\} \subseteq \mathbb{C}^2$. Then Z can be parametrized by $t \mapsto (t^2, t^3)$. The function $f : (t^2, t^3) \mapsto t$ is weakly holomorphic. Outside $Z_{\text{sing}} = \{0\}$, f is equal to w/z, so it is holomorphic on Z_{reg}, and by the formula $(t^2, t^3) \mapsto t$, it is clear that it is locally bounded at $Z_{\text{sing}} = \{0\}$.

However, f is not strongly holomorphic at 0. In fact, a function is weakly holomorphic near the origin on Z if and only if it is holomorphic in t, and it is strongly holomorphic if and only if it is holomorphic in t, and the Taylor expansion in t contains no first order term.

Example 2. Let $Z = \{zw = 0\} \subseteq \mathbb{C}^2$, that is, $Z = V_1 \cup V_2$, where $V_1 = \{z = 0\}$ and $V_2 = \{w = 0\}$. Let

$$f(z) = \begin{cases} -1 & z \in V_1 \setminus \{0\} \\ 1 & z \in V_2 \setminus \{0\} \end{cases}$$

Then f is weakly holomorphic on Z. Note that $f = (z + w)/(z - w)$, and that it is not strongly holomorphic since it is not even continuous at 0.

In both examples, the weakly holomorphic functions are meromorphic, something that is in fact true for all weakly holomorphic functions. This follows from the existence of a so-called universal denominator.

Theorem 1.1. *Let (Z, z) be the germ of an analytic variety. Then there exists a strongly holomorphic function h, not vanishing identically on any irreducible component of (Z, z), such that $h\tilde{\mathcal{O}}_{Z,z} \subseteq \mathcal{O}_{Z,z}$.*

In Example 2, f is not continuous at 0, but it has a continuous extension on each of the components V_1 and V_2 separately. This is in fact the general case, any weakly holomorphic function will have a continuous extension along each irreducible component of Z. The class of functions where these extensions coincide, i.e., continuous weakly holomorphic functions are called *c-holomorphic*. In particular, if the space is locally irreducible, then any weakly holomorphic function is continuous.

An important subclass of analytic spaces are those on which the weakly and strongly holomorphic functions coincide, the so called normal spaces.

Definition 4. Let Z be an analytic space. A point $z \in Z$ is said to be *normal* if $\mathcal{O}_{Z,z} = \tilde{\mathcal{O}}_{Z,z}$, and Z is said to be normal if all its points are normal.

An important tool in the study of weakly holomorphic functions on Z is the so-called normalization of Z, which is a space Y such that the weakly holomorphic functions on Z correspond to the strongly holomorphic functions on Y.

Definition 5. Let Y and Z be analytic spaces, and let $\pi : Y \to Z$ be a holomorphic mapping. Then (Y, π) is a *normalization* of Z if Y is normal and the following conditions hold:
a) π is proper and finite, i.e., the inverse image of a compact set is compact, and the inverse image of a point is a finite set of points.
b) π is biholomorphic outside Z_{sing}, i.e., if $A = \pi^{-1}(Z_{\text{sing}})$, then $\pi|_{Y \setminus A} : Y \setminus A \to Z_{\text{reg}}$ is a biholomorphism.

Theorem 1.2. *Let Z be an analytic space. Then Z has a normalization $\pi : Y \to Z$, which is unique up to isomorphism.*

For a more thorough exposition of the theory of holomorphic functions on analytic spaces, and for proofs of the above theorems, see for example [5] or [8].

2. The Coleff-Herrera product defined by analytic continuation

The main object of study in this thesis is the Coleff-Herrera product, so we will describe in more detail how it can be defined. However, we will mostly work with a definition by Yger based on analytic continuation, different from the original one as in Definition 1. This is based on a result by Atiyah and Bernstein-Gel'fand, that one can do this for principal value current of one holomorphic function. In the most important case, that the functions define a complete intersection, this definition based on analytic continuation and the definition in Definition 1 coincide, see [10]. In the next section, we will only discuss the Coleff-Herrera product of a complete intersection, and hence, one can use either definition. However, in the general case, the definition described below will not coincide with the original Coleff-Herrera product, but there are variations of it which will, see Paper I.

If $f = (f_1, \dots, f_p)$ are holomorphic on $\Omega \subseteq \mathbb{C}^n$, one can define the Coleff-Herrera product of f by

$$\bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} . \varphi := \int \frac{\bar\partial |f_p|^{2\lambda} \wedge \cdots \wedge \bar\partial |f_1|^{2\lambda}}{f_p \cdots f_1} \wedge \varphi \bigg|_{\lambda=0}, \tag{2.1}$$

where the integral on the right-hand side is analytic in λ for $\operatorname{Re} \lambda \gg 0$, and $|_{\lambda=0}$ denotes the analytic continuation to $\lambda = 0$. We will elaborate a bit on how one can prove that this analytic continuation exists.

The basic currents used to build the Coleff-Herrera product (in this way) are the principal value current $1/z^k$ and the residue current $\bar\partial(1/z^k)$ associated to a monomial in $\mathbb{C}$. If φ is a test form, $1/z^k$ is defined by

$$\frac{1}{z^k}.\varphi := \int \frac{|z|^{2\lambda}\varphi}{z^k}\bigg|_{\lambda=0}. \tag{2.2}$$

By a Taylor expansion of the test form, one can check that this analytic continuation to $\lambda = 0$ exists and defines a current. We then define $\bar\partial(1/z^k)$ as $\bar\partial$ of $1/z^k$ in the current sense. It is easy to see that it is equal to

$$\bar\partial\frac{1}{z^k}.\varphi := \int \frac{\bar\partial|z|^{2\lambda}}{z^k} \wedge \varphi\bigg|_{\lambda=0}.$$

This current will coincide with the Coleff-Herrera product as described in (0.1), as both definitions of $\bar\partial(1/z^k)$ give

$$\frac{1}{2\pi i}\bar\partial\frac{1}{z^k}.\varphi(z)dz = \frac{1}{(k-1)!}\partial_z^{k-1}\varphi(0).$$

Assume now we have a tuple $f = (f_1, \dots, f_p)$ of holomorphic functions of the form $f_i = u_i z^{\alpha_i}$, where u_i are non-zero. By using the formula $|z|^{2\lambda}\bar\partial|z|^{2\mu} = (\mu/(\mu+\lambda))\bar\partial|z|^{2(\lambda+\mu)}$, one can verify that the analytic continuation to $\lambda = 0$ of (2.1) exists for this choice of f.

For the general case, we use Hironaka's theorem of resolution of singularities, [1], which says that for any analytic space Z, and tuple $f = (f_1, \dots, f_p)$ of holomorphic functions on Z, there exists a complex manifold $\tilde Z$ and a proper surjective holomorphic mapping $\pi : \tilde Z \to Z$, biholomorphic outside of a nowhere dense analytic subset, such that $\pi^* f$ has normal crossings, i.e., we can choose local coordinates such that $\{\pi^* f = 0\} = \{z_1 \dots z_k = 0\}$. If we consider $f = f_1 \dots f_p$ in Hironaka's theorem, we get locally that $f_1 \dots f_k = u z_1^{\beta_1} \dots z_k^{\beta_k}$, where $u \neq 0$, and hence f_i must be locally in $\tilde Z$ of the form $f_i = u_i z^{\alpha_i}$, where $u_i \neq 0$. Thus, the problem of existence of analytic continuation is reduced to the previous case by taking pull-back to $\tilde Z$.

Note also that the definition (2.1) works equally well if we are on a smooth space, or a singular analytic variety. In that case, the integral on the right-hand side of (2.1) should be interpreted as integration on Z, which is defined as integration on Z_{reg}. This is the current of integration $[Z]$, proved by Lelong to exist. That this defines a current (i.e., has locally finite mass near Z_{sing}) follows from Hironaka's theorem (although it can also be proven by more elementary means).

In Paper I, we will describe a natural generalization of the Coleff-Herrera product of a tuple of weakly holomorphic functions. This has earlier been done by Denkowski for c-holomorphic functions in [6]. The basic idea is to go to the normalization, and hence essentially reduce the problem to the strongly holomorphic case.

3. The duality theorem

Probably the most striking property of the Coleff-Herrera product associated with a complete intersection on a complex manifold is the following theorem due to Dickenstein-Sessa and Passare independently, [7], [11]. A tuple $f = (f_1, \dots, f_p)$ is said to define a *complete intersection* if $\operatorname{codim}\{f_1 = \dots = f_p = 0\} = p$. The *annihilator* of a current μ, denoted $\operatorname{ann}\mu$, is the ideal of holomorphic functions h such that $h\mu = 0$.

Theorem 3.1 (The duality theorem)**.** *Let $f = (f_1, \dots, f_p) \in \mathcal{O}^{\oplus p}(X)$ be a complete intersection on a complex manifold X. Then locally,*

$$\operatorname{ann} \bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1} = (f_1, \dots, f_p).$$

In the case of one single holomorphic function, we can sketch the proof of this. Assume $g\bar\partial(1/f) = 0$. Since $\bar\partial(1/f)$ is $\bar\partial$ of $1/f$ in the current sense, and g is holomorphic, this is equivalent to

$$\bar\partial \left(g \frac{1}{f} \right) = 0.$$

By regularity of the $\bar\partial$-operator on currents, any $\bar\partial$-closed 0-current is in fact a holomorphic function. Thus, $g(1/f)$ is holomorphic, i.e., $g \in (f)$. Conversely, if $g \in (f)$, then $g/f = h \in \mathcal{O}$, and thus

$$g\bar\partial \frac{1}{f} = \bar\partial \left(g \frac{1}{f} \right) = \bar\partial h = 0.$$

If we consider holomorphic functions on a singular variety instead, it is easy to find counterexamples to the duality theorem. In Paper II, we will prove that if the intersection between the zero set of f and certain singularity subvarieties of Z are sufficiently small, the duality theorem holds for the Coleff-Herrera product of f. We will also show that on a singular variety, one can always find a complete intersection such that the duality theorem does not hold.

We will indicate here how to prove this last statement in the case Z is a *reduced complete intersection*, i.e., that there exists $g = (g_1, \dots, g_q)$, where $q = \operatorname{codim} Z$, such that $Z = \{g_1 = \dots = g_q = 0\}$ and $dg_1 \wedge \dots \wedge dg_q$ does not vanish on Z_{reg}. The Poincaré-Lelong formula gives a representation of the integration current $[Z]$ by the following formula:

$$[Z] = \frac{1}{(2\pi i)^q} \bar\partial \frac{1}{g_q} \wedge \dots \wedge \bar\partial \frac{1}{g_1} \wedge dg_1 \wedge \dots \wedge dg_q. \tag{3.1}$$

We now consider a complete intersection $f = (f_1, \dots, f_p)$ on Z, and its associated Coleff-Herrera product, seen as a current in the ambient space, i.e.,

$$\bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1} \wedge [Z]. \tag{3.2}$$

For convenience of notation, we will write $\bar{\partial}(1/f)$ and $\bar{\partial}(1/g)$ for the Coleff-Herrera products of f and g, and $dg = dg_1 \wedge \cdots \wedge dg_q$. By (3.1), we get that (3.2) can be written as

$$\bar{\partial}\frac{1}{f} \wedge [Z] = \frac{1}{(2\pi i)^q}\bar{\partial}\frac{1}{f} \wedge \bar{\partial}\frac{1}{g} \wedge dg. \tag{3.3}$$

In general, one can not multiply currents, but in this situation the multiplication can be justified and the product of $\bar{\partial}(1/f)$ and $\bar{\partial}(1/g)$ in (3.3) will equal the Coleff-Herrera product of (f, g). Hence, if we assume that h annihilates $\bar{\partial}(1/f) \wedge [Z]$, then

$$h \det\left(\frac{\partial g_j}{\partial z_{I_i}}\right) \in \operatorname{ann} \bar{\partial}\frac{1}{f} \wedge \bar{\partial}\frac{1}{g} = (f, g),$$

where $I \subseteq \{1, \ldots, n\}$ and $|I| = q$, by (3.3) and Theorem 3.1. However, if $\{f = 0\} \subseteq Z_{\text{sing}}$, since $dg_1 \wedge \cdots \wedge dg_q$ necessarily vanishes at Z_{sing}, it will not follow that $h \in (f, g)$, i.e., $h \in (f)$ in $\mathcal{O}_Z = \mathcal{O}/(g)$. In fact, as we will show in Paper II, one can always find f and h in this way so that h annihilates the Coleff-Herrera product of f, while $h \notin (f)$.

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RESIDUE CURRENTS ASSOCIATED WITH WEAKLY HOLOMORPHIC FUNCTIONS

RICHARD LÄRKÄNG

Abstract. We construct Coleff-Herrera products and Bochner-Martinelli type residue currents associated with a tuple f of weakly holomorphic functions, and show that these currents satisfy basic properties from the (strongly) holomorphic case. This include the transformation law, the Poincaré-Lelong formula and the equivalence of the Coleff-Herrera product and the Bochner-Martinelli type residue current associated with f when f defines a complete intersection.

1. Introduction

The basic example of a residue current, introduced by Coleff and Herrera in [12], is a current called the *Coleff-Herrera product* associated with a strongly holomorphic mapping $f = (f_1, \dots, f_p)$. The Coleff-Herrera product is defined by

$$\bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1} . \varphi = \lim_{\delta \to 0^+} \int_{\cap \{|f_i| = \epsilon_i(\delta)\}} \frac{\varphi}{f_1 \cdots f_p}, \tag{1.1}$$

where φ is a test form and $\epsilon(\delta)$ tends to 0 along a so-called *admissible path*, which means essentially that $\epsilon_1(\delta)$ tends to 0 much faster than $\epsilon_2(\delta)$ and so on, for the precise definition, see [12]. The Coleff-Herrera product was defined over an analytic space, however, most of the work on residue currents thereafter has focused on the case of holomorphic functions on a complex manifold. The theory of residue currents has various applications, for example to effective versions of division problems etc., see for example [3], [9], [23] and the references therein.

On an analytic space Z, which we throughout the article assume to be of pure dimension, the most common notion of holomorphic functions are the *strongly holomorphic* functions, that is, functions which are locally the restriction of holomorphic functions in any local embedding. In some cases, this can be a little too restrictive, and the *weakly holomorphic* functions might be more natural. These are functions defined on Z_{reg}, which are holomorphic on Z_{reg} and locally bounded at Z_{sing}. Two reasons why these are natural: weakly holomorphic functions are the integral closure of the strongly holomorphic functions in the ring of meromorphic functions, and weakly holomorphic functions correspond to strongly holomorphic functions in any normal modification of Z. A slightly better behaved but more restrictive notion are the *c-holomorphic* functions, functions which are weakly holomorphic and continuous on all of Z. We will use the notation $f \in \mathcal{O}(Z)$ is f is strongly holomorphic on Z, $f \in \tilde{\mathcal{O}}(Z)$ is f is weakly holomorphic on Z, and $f \in \mathcal{O}_c(Z)$ if f is c-holomorphic on Z.

In a recent article [13], Denkowski introduced a residue calculus for c-holomorphic functions, and showed that this calculus satisfies many of the basic properties known from the strongly holomorphic or smooth cases. It is then a natural question to ask what happens in the case of weakly holomorphic functions. However, as in the c-holomorphic case, it is not obvious how to define the associated residue currents.

In the strongly holomorphic case, there are various ways to define the Coleff-Herrera product (for the equivalence of various definitions of the Coleff-Herrera product, also in the non complete intersection case, see for example [19]). The definition we will use is based on analytic continuation as in [25], which was inspired by the ideas in [6] and [7] that the principal value current $1/f$ of a holomorphic function f can be defined by $(|f|^{2\lambda}/f)|_{\lambda=0}$. If $f = (f_1, \dots, f_p)$ is strongly holomorphic on Z, we define the Coleff-Herrera product of f by

$$\left.\frac{\bar\partial|f_p|^{2\lambda_p} \wedge \dots \wedge \bar\partial|f_1|^{2\lambda_1}}{f_1 \cdots f_p}\right|_{\lambda_1=0,\dots,\lambda_p=0}$$

where we by $|_{\lambda_1=0,\dots,\lambda_p=0}$ mean that we take the analytic continuation in λ_1 to $\lambda_1 = 0$, then in λ_2 and so on, see Section 4 for details. Recall that a *modification* of an analytic space Z is a proper surjective holomorphic mapping $\pi : Y \to Z$ from an analytic space Y such that there exists a nowhere dense analytic set $E \subset X$ with $\pi|_{Y\setminus\pi^{-1}(E)} : Y \setminus \pi^{-1}(E) \to X \setminus E$ a biholomorphism. It is easy to see by analytic continuation, that if $\pi : Y \to Z$ is a modification of Z, then the Coleff-Herrera product of f can be defined as the push-forward of the Coleff-Herrera product of $f' := \pi^* f$. For weakly holomorphic functions, we can use this observation to define the Coleff-Herrera product, since the pull-back of a weakly holomorphic function to the normalization is strongly holomorphic. If f is weakly holomorphic, we define the Coleff-Herrera product of f by

$$\mu^f := \bar\partial\frac{1}{f_p} \wedge \dots \wedge \bar\partial\frac{1}{f_1} := \pi_*\left(\bar\partial\frac{1}{f'_p} \wedge \dots \wedge \bar\partial\frac{1}{f'_1}\right), \tag{1.2}$$

where $f' = \pi^* f$. By the observation above, this of course coincides with the usual definition in case of strongly holomorphic functions, and this definition is also consistent with the one in [13] in the case of c-holomorphic functions, see Proposition 4.1.

Because of our definition, the properties we prove of the Coleff-Herrera product for weakly holomorphic functions can mostly be reduced (by going back to the normalization) to the strongly holomorphic case. Thus the main part of this article concerns giving a coherent exposition of the basic theory of residue currents in the strongly holomorphic case. This is done based on analytic continuation of currents and the notion of pseudomeromorphic currents as introduced in [4], which is developed on a complex manifold. We will see that this approach works well also with strongly holomorphic functions on an analytic space, and we believe that this might be of independent interest, although most of the results should be known.

However, even for the statement of these properties in the weakly holomorphic case, two problems occur, namely how is multiplication of a weakly holomorphic function with a current defined, and what is the zero set of a tuple of weakly holomorphic functions? And hence also, what should a complete intersection mean?

For the problem of multiplication of weakly holomorphic functions with currents, we take a similar approach as for the definition of the Coleff-Herrera product. Namely, if $\pi : Y \to Z$ is a modification, μ is a current on Z, g is strongly holomorphic on Z and $\mu = \pi_* \mu'$, then

$$g\mu = \pi_*(\pi^* g\mu'). \tag{1.3}$$

The right-hand side of (1.3) still exists if g is weakly holomorphic on Z and Y is normal, but as we will see in Section 5, it will in general depend on the choice of representative μ'. However, if μ is a Coleff-Herrera product, then we have a certain "canonical" representative μ' of μ in the normalization, and we define $g\mu$ by (1.3) with this choice of μ'.

For the zero set of one weakly holomorphic function, all reasonable definitions should coincide. For the zero set of a weakly holomorphic mapping f, it is natural to take into account that the zero sets of the individual components of f can "belong" to different irreducible components. We introduce in Section 2 a notion of common zero set of f, depending on f as a mapping, and not only on the individual components, which however may differ from the intersection of the respective zero sets.

The Coleff-Herrera product μ^f in (1.2) associated with a strongly holomorphic mapping $f = (f_1, \dots, f_p)$ satisfies

$$\operatorname{supp} \mu^f \subseteq Z_f \quad \text{and} \quad \bar\partial \mu^f = 0,$$

where Z_f is the common zero set of f. In addition, if f forms a complete intersection, the Coleff-Herrera product is alternating in the residue factors and

$$(f_1, \dots, f_p) \subseteq \operatorname{ann} \mu^f, \tag{1.4}$$

where $(f_1, \dots, f_p)$ is the ideal generated by $f_1, \dots, f_p$, and $\operatorname{ann} \mu^f$ is the annihilator of μ^f, i.e., the ideal of holomorphic functions g such that $g\mu^f = 0$. We also have the *transformation law* for residue currents (see [14]), which says that if $f = (f_1, \dots, f_p)$ and $g = (g_1, \dots, g_p)$ define a complete intersection, and there exists a matrix A of holomorphic functions such that $g = Af$, then

$$(\det A)\bar\partial\frac{1}{g_p}\wedge\cdots\wedge\bar\partial\frac{1}{g_1} = \bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.$$

The *Poincaré-Lelong formula* relates the Coleff-Herrera product of f and the integration current $[Z_f]$ on Z_f (with multiplicities) and it says that

$$\frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge df_1\wedge\cdots\wedge df_p = [Z_f].$$

We will see that in fact all those statements still hold also in the weakly holomorphic case. However, as mentioned above, zero sets of weakly holomorphic functions and multiplication of currents with weakly holomorphic functions need to be interpreted in the right way.

Remark 1. The inclusion (1.4) if f defines a complete intersection is one direction of the *duality theorem* proven in [14] and [21], which says that on a complex manifold, the inclusion is in fact (locally) an equality. However, in [18], we show that on any singular variety, one can always find a tuple f of strongly holomorphic functions such that the inclusion (1.4) is strict.

Bochner-Martinelli type residue currents were first introduced in [22] by Passare, Tsikh and Yger (on a complex manifold) as an alternative way of defining a residue current corresponding to a tuple of holomorphic functions. In [10], Bochner-Martinelli type residue currents were constructed on an analytic space in order to prove a generalization of Jacobi's residue formula, generalizing previous results in [24] in the smooth case.

The Bochner-Martinelli type residue currents give another reason why our definition of Coleff-Herrera product is a natural one. In the smooth case, it was proved in [22] that if the functions define a complete intersection, then the Coleff-Herrera product and the Bochner-Martinelli current coincide. It is suggested in [10] that the same statement holds in the singular case with a similar proof. We will construct Bochner-Martinelli type residue currents associated with a tuple of weakly holomorphic functions, and we will show that the equality between the Coleff-Herrera product and the Bochner-Martinelli type residue current holds both in the strongly and weakly holomorphic cases. An advantage of the Bochner-Martinelli current in the weakly holomorphic case is that it can be defined intrinsically on Z as the analytic continuation of an arbitrarily smooth (depending on a parameter λ) form on Z.

2. Zero sets of weakly holomorphic functions

The behavior of the currents we define will depend in a crucial way on the zero sets of the weakly holomorphic functions, and in this section we will define the zero set of a weakly holomorphic mapping.

Definition 1. Let $f \in \tilde{\mathcal{O}}(Z)$. If f is not identically zero on any irreducible component of Z, we define the *zero set* of f by $Z_f := \{z \in Z \mid (1/f)_z \notin \tilde{\mathcal{O}}_z\}$. Let Z_α be the irreducible components of Z where f is identically zero, and let $Z' = \overline{Z \setminus \cup_\alpha Z_\alpha}$. Then f does not vanish identically on any of the irreducible components of Z', and we define $Z_f := \cup_\alpha Z_\alpha \cup Z_{f|_{Z'}}$.

For any meromorphic function ϕ, there is a standard notion of zero set of ϕ, that we denote by Z'_ϕ, which is defined by $Z'_\phi := \{z \in Z \mid (1/\phi)_z \notin \mathcal{O}_z\}$. Since weakly holomorphic functions are meromorphic, this gives another definition of zero set if f is a weakly holomorphic function. Clearly $Z_f \subseteq Z'_f$, but note that in general the inclusion can be strict, so the two definitions do not coincide.

Remark 2. We have $z \in Z_f$ if and only if there exists a sequence $z_i \to z$ with $z_i \in Z_{\text{reg}}$ such that $f(z_i) \to 0$ (since if we cannot find such a sequence, then $1/f$ is weakly holomorphic). Hence, when f is c-holomorphic, Z_f coincides with the usual zero set of f, when f is seen as continuous function.

We will use the following characterization of the zero set of a weakly holomorphic function. However, since this is a special case of Proposition 2.3, we omit the proof.

Lemma 2.1. *Let $\pi : Z' \to Z$ be the normalization of Z. If $f \in \tilde{\mathcal{O}}(Z)$, then Z_f is an analytic subset of Z, and $Z_f = \pi(Z_{\pi^* f})$.*

We recall that an analytic space Z is *normal* if $\mathcal{O}_{Z,z} = \tilde{\mathcal{O}}_{Z,z}$ for all $z \in Z$, and that the *normalization* Z' of an analytic space Z is the unique normal space Z' together with a proper finite surjective holomorphic mapping $\pi : Z' \to Z$ such that $\pi|_{Z' \setminus \pi^{-1}(Z_{\text{sing}})} : Z' \setminus \pi^{-1}(Z_{\text{sing}}) \to Z_{\text{reg}}$ is a biholomorphism, see for example [15].

To study the dimension of zero sets of weakly holomorphic functions, we will need the following lemma, which shows that subvarieties of the normalization correspond to subvarieties of Z of the same dimension, and vice versa.

Lemma 2.2. *Let $\pi : Z' \to Z$ be the normalization of Z. If Y' is a subvariety of Z', then $\pi(Y')$ is a subvariety of Z with $\dim Y' = \dim \pi(Y')$, and if Y is a subvariety of Z, then $\pi^{-1}(Y)$ is a subvariety of Z' with $\dim Y = \dim \pi^{-1}(Y)$.*

Proof. The first part follows from Remmert's proper mapping theorem, when formulated as for example in [15], since π is a finite proper holomorphic mapping. We get from the first part that $\dim \pi^{-1}(Y) = \dim \pi(\pi^{-1}(Y)) = \dim Y$, where the second equality holds since π is surjective. $\square$

If $f \in \tilde{\mathcal{O}}(Z)$ and $f \not\equiv 0$ on any irreducible component of Z, then $\operatorname{codim} Z_f = 1$ or $Z_f = \emptyset$. In fact, if $f' = \pi^* f$ and $Z_{f'} \neq \emptyset$, then f' is strongly holomorphic, and $Z_{f'} = \{f' = 0\}$ has codimension 1, and since $Z_f = \pi(Z_{f'})$ by Lemma 2.1, Z_f has codimension 1 by Lemma 2.2. However, as is well-known, in contrast to the smooth case, subvarieties of codimension 1 cannot in general be defined as the zero set of one single strongly holomorphic function. As we will see in the next example, this is the case in general for zero sets of weakly holomorphic functions, even for c-holomorphic functions on an irreducible space.

Example 1. Let $V = \{z_1^3 - z_2^2 = z_3^3 - z_4^2 = 0\} \subset \mathbb{C}^4$. Then V has normalization $\pi : \mathbb{C}^2 \to V$, $\pi(t_1, t_2) = (t_1^2, t_1^3, t_2^2, t_2^3)$, and hence $f = z_2/z_1 - z_4/z_3$ is c-holomorphic since $\pi^* f = t_1 - t_2$. The set $Z_f = \{(t^2, t^3, t^2, t^3)\}$ has codimension 1 in Z. However, there does not exist a holomorphic function in a neighborhood of 0 such that $f(t_1^2, t_1^3, t_2^2, t_2^3) = 0$ exactly when $t_1 = t_2$, since in that case, we could write $f(t_1^2, t_1^3, t_2^2, t_2^3) = (t_1 - t_2)^m u(t_1, t_2)$ for some $m \in \mathbb{N}$, where $u(0,0) \neq 0$, which is easily seen to be impossible. Hence, Z_f is not the zero set of one single strongly holomorphic function.

Example 2. Let $Z = Z_1 \cup Z_2 \subset \mathbb{C}^6$, where $Z_1 = \mathbb{C}^3 \times \{0\}$ and $Z_2 = \{0\} \times \mathbb{C}^3$. Define the functions f and g by

$$f(z) = \begin{cases} z_1 & z \in Z_1 \setminus \{0\} \\ 1 & z \in Z_2 \setminus \{0\} \end{cases} \text{ and } g(z) = \begin{cases} 1 & z \in Z_1 \setminus \{0\} \\ z_4 & z \in Z_2 \setminus \{0\} \end{cases}.$$

Then $f, g \in \tilde{\mathcal{O}}(Z)$, and $Z_f = Z_1 \cap \{z_1 = 0\}$, and $Z_g = Z_2 \cap \{z_4 = 0\}$ which both have codimension 1 in Z. However, $Z_f \cap Z_g = \{0\}$, which has codimension 3. Hence, zero sets of weakly holomorphic functions do not behave as well as one could hope with respect to intersections. If we let $f_1 = f_2 = f$, $f_3 = g$, then $Z_{f_1} \cap Z_{f_2} \cap Z_{f_3} = \{0\}$ has codimension 3, while $Z_{f_1} \cap Z_{f_2} = Z_f$ has codimension 1 at 0 in Z. Hence, if one defines a complete intersection for zero sets of weakly holomorphic functions $f = (f_1, \cdots, f_p)$ by requiring that $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension p in Z, then it will not follow in general that $(Z_{f_1} \cap \cdots \cap Z_{f_k}, z)$ has codimension k for $z \in Z_{f_1} \cap \cdots \cap Z_{f_p}$.

Remark 3. Note that for c-holomorphic functions $f = (f_1, \cdots, f_p)$, if $f' = \pi^* f$, where $\pi : Z' \to Z$ is the normalization, then $\pi(Z_{f_1'} \cap \cdots \cap Z_{f_p'}) = Z_{f_1} \cap \cdots \cap Z_{f_p}$. Thus if we say that $f = (f_1, \cdots, f_p)$, where $f_i \in \mathcal{O}_c(Z)$, forms a complete intersection in Z if $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension p, then this holds if and only if f' forms a complete intersection in Z' by Lemma 2.2.

As we see in Example 2, this remark does not hold for weakly holomorphic functions, because there, $Z_f \cap Z_g = \{0\}$, while $Z_{f'} \cap Z_{g'} = \emptyset$. Thus, the straight forward generalization of complete intersection, where the zero set $Z_{f_1} \cap \cdots \cap Z_{f_p}$ is required to have codimension p does not share the same good properties in the weakly holomorphic case as in the strongly holomorphic (or c-holomorphic) case. Because of this, we will use a different definition of both the common zero set of weakly holomorphic functions and of a complete intersection. However, it coincides with the usual definitions in case of strongly holomorphic or c-holomorphic functions, and with our definition the problems above disappear.

Definition 2. Let $f = (f_1, \cdots, f_p)$ be weakly holomorphic. We define the *common zero set* of f, denoted by Z_f, as the set of $z \in Z$ such that there exists a sequence $z_i \in Z_{\text{reg}}$ with $z_i \to z$, and $f_k(z_i) \to 0$ for $k = 1, \cdots, p$. We will see that Z_f is an analytic subset of Z, and hence we say that f forms a *complete intersection* if Z_f has codimension p in Z.

Note that by Remark 2, this definition is consistent with the definition of Z_f in the case of one function. We also see that in Example 2, $Z_{(f,g)} = \emptyset$, and hence, (f, g) is not a complete intersection in our sense. Just as for one function, we can give a characterization of the zero set with the help of the normalization.

Proposition 2.3. *Let $f = (f_1, \cdots, f_p)$ be weakly holomorphic, and let $f' = \pi^* f$, where $\pi : Z' \to Z$ is the normalization. Then*

$$Z_f = \pi(Z_{f_1'} \cap \cdots \cap Z_{f_p'}), \tag{2.1}$$

and Z_f is an analytic subset of Z of codimension $\leq p$. In general,

$$Z_f \subseteq Z_{f_1} \cap \cdots \cap Z_{f_p}, \tag{2.2}$$

with equality if f is c-holomorphic. In addition, f is a complete intersection if and only if f' is a complete intersection in the normalization.

Proof. If $z' \in Z_{f_1'} \cap \cdots \cap Z_{f_p'}$, then we can take a sequence $z_i' \to z'$ such that $z_i' \in \pi^{-1}(Z_{\text{reg}})$. Then, if we let $z_i = \pi(z_i')$, we get that $f_k(z_i) \to 0$, and hence we have the inclusion $Z_f \supseteq \pi(Z_{f_1'} \cap \cdots \cap Z_{f_p'})$ in (2.1). For the other inclusion, if we have a sequence $z_i \to z$ such that $z \in Z_f$, since π is proper we can choose a convergent subsequence $z_{k_i}' \to z'$ such that $\pi(z_{k_i}') = z_{k_i}$, and since $z \in Z_f$, we must have $f'(z') = 0$, so $z = \pi(z')$, with $z' \in Z_{f_1'} \cap \cdots \cap Z_{f_p'}$. Now, the fact that Z_f is an analytic subset of Z follows by (2.1) and Remmert's proper mapping theorem, since $Z_{f_i'}$ are analytic subsets of Z'. Since f' is strongly holomorphic, $Z_{f'}$ has codimension $\leq p$, so by (2.1) combined with Lemma 2.2 we get that Z_f has codimension $\leq p$. If f is c-holomorphic, the equality in (2.2) follows by (2.1) since for any continuous mapping

f, $Z_{f_1}\cap\cdots\cap Z_{f_p} = \pi(Z_{\pi^* f_1}\cap\cdots\cap Z_{\pi^* f_p})$, and the general case also follows from (2.1) since $\pi(Z_{f'_1}\cap\cdots\cap Z_{f'_p}) \subseteq \pi(Z_{f'_1})\cap\cdots\cap\pi(Z_{f'_p}) = Z_{f_1}\cap\cdots\cap Z_{f_p}$. Finally, the fact that f is a complete intersection if and only if f' is a complete intersection follows from (2.1) together with Lemma 2.2. $\square$

We note that if $Z_{f_1}\cap\cdots\cap Z_{f_p}$ has codimension $\geq p$, then either $Z_f = \emptyset$, or Z_f has codimension p since by Proposition 2.3, $Z_f \subseteq Z_{f_1}\cap\cdots\cap Z_{f_p}$, and Z_f has codimension at most p. Thus, if $Z_{f_1}\cap\cdots\cap Z_{f_p}$ has codimension $\geq p$, and some result depends on the fact that Z_f should have codimension $\geq p$, it will still be true with this other definition of complete intersection. This will be the case for all results about residue currents stated here, except for the Poincaré-Lelong formula, Proposition 8.1. Hence, our definition of complete intersection, Definition 2 is not essential for the results to hold, however, the requirement that $Z_{f_1}\cap\cdots\cap Z_{f_p}$ should have codimension $\geq p$ will in general give weaker statements, since it might very well happen that $Z_{f_1}\cap\cdots\cap Z_{f_p}$ has codimension $< p$, while Z_f has codimension p.

Note also that, if $f = (f_1, \cdots, f_p)$ is a complete intersection and $f_0 = (f_1, \cdots, f_k)$, then (Z_{f_0}, z) has codimension k for $z \in Z_f$, since if $z' \in \pi^{-1}(z)$, then $(Z_{f'_0}, z')$ has codimension k, and hence since π is a finite proper holomorphic mapping, $(Z_{f_0}, z) = \cup_{z'_j\in\pi^{-1}(z)}\pi((Z_{f'_0}, z'_j))$ has codimension k in Z.

3. Pseudomeromorphic currents on an analytic space

We will in this section introduce pseudomeromorphic currents on an analytic space. Pseudomeromorphic currents on a complex manifold were introduced by Andersson and Wulcan in [4], inspired by the fact that currents like the Coleff-Herrera product and Bochner-Martinelli type residue currents are pseudomeromorphic. Two important properties of pseudomeromorphic currents in the smooth case are the direct analogues of Proposition 3.1 and Proposition 3.2. Since these hold also in the singular case, many properties of residue currents hold also for strongly holomorphic functions by more or less the same argument as in the smooth case.

The pseudomeromorphic currents are intrinsic objects of the analytic space Z, so we begin with explaining what we mean by a current on an analytic space. We will follow the definitions used in [8] and [16]. To begin with, we assume that Z is an analytic subvariety of Ω, for some open set $\Omega \subseteq \mathbb{C}^n$. Then, we define the set of smooth forms of bidegree (p, q) in Z by $\mathcal{E}_{p,q}(Z) = \mathcal{E}_{p,q}(\Omega)/\mathcal{N}_{p,q,Z}(\Omega)$, where $\mathcal{E}_{p,q}(\Omega)$ are the smooth (p, q)-forms in Ω and $\mathcal{N}_{p,q,Z}(\Omega) \subset \mathcal{E}_{p,q}(\Omega)$ are the smooth forms φ such that $i^*\varphi \equiv 0$, where $i : Z_{\text{reg}} \to \Omega$ is the inclusion map. The set of test forms on Z, $\mathcal{D}_{p,q}(Z)$, are the forms in $\mathcal{E}_{p,q}(Z)$ with compact support. With the usual topology on $\mathcal{D}_{p,q}(\Omega)$ by uniform convergence of coefficients of differential forms together with their derivatives on compact sets, we give $\mathcal{D}_{p,q}(Z)$ the quotient topology from the projection $\mathcal{D}_{p,q}(\Omega) \to \mathcal{D}_{p,q}(Z)$. Then, (p,q)-currents on Z, denoted $\mathcal{D}'_{p,q}$, are the continuous linear functionals on $\mathcal{D}_{k-p,k-q}(Z)$, where $k = \dim Z$. However, more concretely, this just means that if μ is a (p,q)-current on Z, then $i_*\mu$ is a $(n-k+p, n-k+q)$-current in the usual sense on Ω that vanishes on forms in $\mathcal{N}_{k-p,k-q,Z}(\Omega)$. Conversely, if T is a $(n-k+p, n-k+q)$-current on Ω, that vanishes on forms in $\mathcal{N}_{k-p,k-q,Z}(\Omega)$, then T defines a unique (p,q)-current T' on Z such that $i_*T' = T$.

It is easy to see that the definitions of smooth forms, test forms and currents are independent of the embedding, and hence by gluing together in the same way one does on a complex manifold, we can define the sheafs of smooth forms, test forms and currents on any analytic space Z. Note in particular that by a smooth function on Z, we mean a function which is locally the restriction of a smooth function in the ambient space.

In $\mathbb{C}$, one can define the principal value current $1/z^n = |z|^{2\lambda}/z^n|_{\lambda=0}$ by analytic continuation, where $|_{\lambda=0}$ denotes that for $\operatorname{Re}\lambda \gg 0$, we take the action of $|z|^{2\lambda}/z^n$ on a test-form and take the value of the analytic continuation to $\lambda = 0$, which is easily seen to exist by a Taylor expansion, or integration by parts. Thus, if α is a smooth form on $\mathbb{C}^n$ and $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}$, with i_j disjoint, then one gets a well-defined current

$$\frac{1}{z_{i_1}^{n_1}} \cdots \frac{1}{z_{i_k}^{n_k}} \bar\partial \frac{1}{z_{i_{k+1}}^{n_{k+1}}} \wedge \cdots \wedge \bar\partial \frac{1}{z_{i_m}^{n_m}} \wedge \alpha \tag{3.1}$$

on $\mathbb{C}^n$ by taking $\bar\partial$ in the current sense together with tensor product of currents and multiplication of currents with smooth forms. In [4], if α has compact support, a current of the form (3.1) is called an *elementary current*. The class of pseudomeromorphic currents on a complex manifold was then introduced as currents that can be written as a locally finite sum of push-forwards of elementary currents. We will use the same definition on an analytic space Z.

Definition 3. Let $\pi_\alpha : Z_\alpha \to Z$ be a family of modifications of Z, where Z_α are complex manifolds. The class of *pseudomeromorphic currents*, denoted $\mathcal{PM}(Z)$ are the currents μ on Z that can be written as a locally finite sum

$$\mu = \sum (\pi_\alpha)_* \tau_\alpha,$$

where τ_α are elementary currents on Z_α.

Note in particular that, if $\pi : \tilde Z \to Z$ is a resolution of singularities of Z, and if $\mu \in \mathcal{PM}(\tilde Z)$, then $\pi_*\mu \in \mathcal{PM}(Z)$. All the currents introduced in this article are pseudomeromorphic, as we will see directly from the proofs that the currents exist. In [4], it is shown that if f is holomorphic on a complex manifold X, and $T \in \mathcal{PM}(X)$, one can define a multiplication $(1/f)T$ and $\bar\partial(1/f)\wedge T$. The same idea works equally well for strongly holomorphic functions on an analytic space.

Proposition 3.1. *Let f be strongly holomorphic on Z and $T\in\mathcal{PM}(Z)$. Then the currents*

$$\frac{1}{f}T := \frac{|f|^{2\lambda}}{f}T\bigg|_{\lambda=0} \quad \text{and} \quad \bar\partial\frac{1}{f}\wedge T := \frac{\bar\partial|f|^{2\lambda}}{f}\wedge T\bigg|_{\lambda=0},$$

where the right-hand sides are defined originally for $\operatorname{Re}\lambda\gg 0$, have current-valued analytic continuations to $\operatorname{Re}\lambda > -\epsilon$ for some $\epsilon>0$, and the values at $\lambda=0$ are pseudomeromorphic. The currents satisfies the Leibniz rule

$$\bar\partial\left(\frac{1}{f}T\right) = \bar\partial\frac{1}{f}\wedge T + \frac{1}{f}\bar\partial T,$$

and $\operatorname{supp}(\bar\partial(1/f)\wedge T)\subseteq Z_f\cap\operatorname{supp}T$. If $f\neq 0$, then $(1/f)T$ defined in this way coincides with the usual multiplication of T with the smooth function $1/f$.

Proof. If Z is smooth, this is Proposition 2.1 in [4], except for the last statement. However, if $f\neq 0$, then $|f(z)|^{2\lambda}/f(z)$ is smooth in both λ and z, and analytic in λ, so if ξ is a test form, $T.((|f|^{2\lambda}/f)\xi)$ is analytic in λ, and hence the analytic continuation to $\lambda=0$ coincides with the value $T.((1/f)\xi)$ at $\lambda=0$. The proof in the general case goes through word for word as in the smooth case in Proposition 2.1 in [4]. $\square$

The crucial point in the proof of the following proposition is that for any analytic subset $W\subseteq Z$ and any $T\in\mathcal{PM}(Z)$, there exist natural restrictions

$$\mathbf{1}_{W^c}T := |h|^{2\lambda}T|_{\lambda=0} \quad \text{and} \quad \mathbf{1}_W T := T - \mathbf{1}_{W^c}T \tag{3.2}$$

where h is a tuple of holomorphic functions such that $W=\{h=0\}$. The restrictions are independent of the choice of such h, and are such that $\operatorname{supp}\mathbf{1}_W T\subseteq W$. This is Proposition 2.2 in [4], and the proof will go through in exactly the same way when Z is an analytic space.

Proposition 3.2. *Assume that $\mu\in\mathcal{PM}(Z)$, and that μ has support on a variety V. If I_V is the ideal of holomorphic functions vanishing on V, then $\bar I_V\mu=0$. If μ is of bidegree $(*,p)$, and V has codimension $\geq p+1$ in Z, then $\mu=0$.*

In the case that Z is a complex manifold, this is Proposition 2.3 and Corollary 2.4 in [4], and the proof there will go through in the same way also when Z is an analytic space. The final step in the proof that $\mu=0$ in the smooth case is to prove that $\mu=0$ on $V_{\rm reg}$, which is proved with the help of the previous part of the proposition, and by degree reasons, and then by induction over the dimension of V, $\mu=0$. In the singular case, this is done in the same way. Since this is a local statement, we can assume that $Z\subseteq\Omega\subseteq\mathbb{C}^n$, and consider V as a subvariety of Ω.

Then, for the same reasons as in the smooth case, we get that $i_*\mu = 0$ on V_{reg}, and by induction over the dimension of V that $i_*\mu = 0$, and hence $\mu = 0$.

4. Coleff-Herrera products of weakly holomorphic functions

Let $f_1, \cdots, f_m \in \tilde{\mathcal{O}}(Z)$. We want to define the Coleff-Herrera product

$$T = \frac{1}{f_m} \cdots \frac{1}{f_{p+1}} \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1}.$$

If f is strongly holomorphic, one way to define it is by

$$T = \frac{|f_m|^{2\lambda_m} \cdots |f_{p+1}|^{2\lambda_{p+1}}}{f_m \cdots f_{p+1}} \frac{\bar{\partial}|f_p|^{2\lambda_p} \wedge \cdots \wedge \bar{\partial}|f_1|^{2\lambda_1}}{f_p \cdots f_1}\bigg|_{\lambda_1=0,\dots,\lambda_m=0}, \tag{4.1}$$

which a priori is defined only when $\operatorname{Re}\lambda_i \gg 0$; however, by Proposition 3.1 it has an analytic continuation in λ_1 to $\operatorname{Re}\lambda_1 > -\epsilon$ for some $\epsilon > 0$, and the value at $\lambda_1 = 0$ is pseudomeromorphic. Again, by Proposition 3.1, it has an analytic continuation in λ_2 to $\lambda_2 = 0$ and so on, and hence the value at $\lambda_1 = 0, \cdots, \lambda_m = 0$ exists. Note that if $\pi : Y \to Z$ is any modification of Z, we can define the corresponding Coleff-Herrera product of $f' = \pi^* f$ in Y, and then take the push-forward to Z, and this will give the same current by analytic continuation.

Now, if f is weakly holomorphic, let $\pi : Z' \to Z$ be the normalization of Z, and $f' = \pi^* f$ which is strongly holomorphic on Z'. Hence, the current

$$T' = \frac{1}{f'_m} \cdots \frac{1}{f'_{p+1}} \bar{\partial}\frac{1}{f'_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f'_1} \tag{4.2}$$

exists.

Definition 4. If $f = (f_1, \dots, f_m)$ is weakly holomorphic, we define the *Coleff-Herrera product* of f by

$$T = \frac{1}{f_m} \cdots \frac{1}{f_{p+1}} \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1} = \pi_* T', \tag{4.3}$$

where T' is defined by (4.2).

If f is strongly holomorphic, this definition will be the same as the definition in (4.1) since by the remark above, T can be defined as the push-forward from any modification. In addition, if f is weakly holomorphic, it can be defined by the push-forward of the corresponding current in any normal modification, since any normal modification factors through the normalization.

We will call the factors $1/f_i$ the *principal value factors*, and $\bar{\partial}(1/f_i)$ the *residue factors*. Note that even though here, the principal value factors are to the left of the residue factors, we could equally well have the residue and principal value factors mixed. However, changing the order will in general give a different current, but as we will see in Theorem 4.3, if f_i define a complete intersection, the current will not depend on the order (up to change of signs).

Remark 4. The Coleff-Herrera product for $f = (f_1, \dots, f_p)$ strongly holomorphic is originally defined in [12] as the limit of integrals over $\cap\{|f_i| = \epsilon_i(\delta)\}$ as $\epsilon \to 0$, where $\epsilon(\delta)$ tends to 0 along an admissible path, cf., (1.1). When $\epsilon(\delta)$ tends to 0 along an admissible path, this will correspond to taking the analytic continuation to $\lambda = 0$ in the order as in (4.1), and in fact, for arbitrary f, the definition in (1.1) is equal to the one in (4.3) defined by analytic continuation, see [19].

In [13] Denkowski gave a definition of the Coleff-Herrera product of f, for f c-holomorphic, and we will see below that his definition coincides with ours in that case. The idea in [13] was to consider the graph of f, $\Gamma_f = \{(z, f(z)) \in Z \times \mathbb{C}^p_w | z \in Z\}$, and even though f is only c-holomorphic, the graph will be analytic. If $(z, w) \in \Gamma_f$, then $w = f(z)$, and hence on the graph $f_i = w_i$ is a strongly holomorphic function. If Π is the projection from the graph to Z, since f is continuous, Π is a homeomorphism and in particular proper. The Coleff-Herrera product of f was then defined by

$$\bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} = \Pi_*\left(\bar\partial\frac{1}{w_p} \wedge \cdots \wedge \bar\partial\frac{1}{w_1}\right), \tag{4.4}$$

and since $f_i = w_i$ on Γ_f, this should be a reasonable definition of the Coleff-Herrera product of f. The next proposition shows, as one might hope, that the definition of Denkowski coincides with ours.

Proposition 4.1. *If $f = (f_1, \dots, f_p)$ is c-holomorphic, then the definition of the Coleff-Herrera product of f in* (4.3) *and in* (4.4) *coincide.*

Proof. In [13] the definition used for the Coleff-Herrera product of strongly holomorphic functions was the one from [12]. However, by Remark 4 we can assume that the definition by analytic continuation is used instead. Let $\pi : Z' \to Z$ be the normalization of Z and $f' = \pi^* f$. We have projections $\Pi : \Gamma_f \to Z$ and $\Pi' : \Gamma_{f'} \to Z'$, where $\Gamma_f \subseteq Z \times \mathbb{C}^p_w$ and $\Gamma_{f'} \subseteq Z' \times \mathbb{C}^p_{w'}$ are the graphs of f and f'. Thus we have a commutative diagram

$$\begin{array}{ccc} \Gamma_{f'} & \xrightarrow{(\pi\times \mathrm{Id})|_{\Gamma_{f'}}} & \Gamma_f \\ \big\downarrow{\scriptstyle \Pi'} & & \big\downarrow{\scriptstyle \Pi} \\ Z' & \xrightarrow{\pi} & Z. \end{array} \tag{4.5}$$

We will denote the current $\bar\partial(1/f'_p) \wedge \cdots \wedge \bar\partial(1/f'_1)$ on Z' by $\mu^{f'}$, and similarly for μ^w and $\mu^{w'}$ defined on Γ_f and $\Gamma_{f'}$ respectively. Then $\bar\partial(1/f_p) \wedge \cdots \wedge \bar\partial(1/f_1)$ is defined in (4.3) by $\pi_*\mu^{f'}$, and in (4.4) by $\Pi_*\mu^w$. Now, $(\pi \times \mathrm{Id})|_{\Gamma_{f'}} : \Gamma_{f'} \to \Gamma_f$ is a modification of Γ_f so we have $\mu^w = (\Pi \times \mathrm{Id})_* \mu^{w'}$, and since $\Pi' : \Gamma_{f'} \to Z'$ is a biholomorphism and $w_i' = f_i'$ on $\Gamma_{f'}$ we also have $\mu^{f'} = \Pi'_* \mu^{w'}$. Thus both are the push-forward of the same current in $\Gamma_{f'}$, and since the diagram (4.5) commutes, both will have the same push-forward to Z. □

The next theorems show that the Coleff-Herrera product of weakly holomorphic functions has some properties that are well-known for strongly holomorphic functions on an analytic space (in the case $m = p$ or $m = p+1$), see [12], or the case of holomorphic functions on a complex manifold, see [20].

Theorem 4.2. *If $f = (f_1, \cdots, f_m)$ is weakly holomorphic, then T, defined by* (4.3)*, satisfies the Leibniz rule*

$$\bar\partial T = \sum_{j=p+1}^{m} (-1)^{m-j} \frac{1}{f_m} \cdots \wedge \bar\partial \frac{1}{f_j} \wedge \cdots \frac{1}{f_{p+1}} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1},$$

and $\operatorname{supp} T \subseteq Z_{(f_1, \cdots, f_p)}$.

Proof. First we assume that f is strongly holomorphic. Then the Leibniz rule follows by analytic continuation, since if $\operatorname{Re} \lambda \gg 0$, we have

$$\bar\partial \left(\frac{|f|^{2\lambda}}{f} \right) = \frac{\bar\partial |f|^{2\lambda}}{f} \text{ and } \bar\partial \left(\frac{\bar\partial |f|^{2\lambda}}{f} \right) = 0.$$

The weakly holomorphic case follows by taking push-forward from the normalization. For the last part, let T' be the current corresponding to T in the normalization, and $f' = \pi^* f$ be the pull-back of f to the normalization. Then by Proposition 3.1, $T' = 0$ outside of $Z_{f_i'}$, and hence $\operatorname{supp} T \subseteq \pi(\operatorname{supp} T') \subseteq \pi(Z_{(f_1', \cdots, f_p')}) = Z_{(f_1, \cdots, f_p)}$, where the last equality follows from Proposition 2.3. □

It is natural in this context to ask how to define a reasonable multiplication of a weakly holomorphic function with a current, something which we will need in the case that the current is a Coleff-Herrera product to be able to state the next theorem. If $g \in \tilde{\mathcal{O}}(Z)$, and T is the Coleff-Herrera product in (4.3), we define gT by

$$gT = \pi_*(\pi^* g T'), \tag{4.6}$$

where $\pi : Z' \to Z$ is the normalization of Z, and T' is the corresponding Coleff-Herrera product of $f' = \pi^* f$. In the case that both f and g are c-holomorphic, Denkowski gives a definition of multiplication of g and the Coleff-Herrera product of f in [13], and by a similar argument as that in Proposition 4.1, one sees that our definition coincides with the one in [13] in that case. Note however, that we do not define a multiplication of a weakly holomorphic function with an arbitrary current, and as we will see in Section 5, this will not be possible if we require it to satisfy certain natural properties.

Theorem 4.3. *Let $f = (f_1, \cdots, f_m)$ be weakly holomorphic, such that $(f_1, \dots, f_p)$ defines a complete intersection, and that $(f_1, \dots, f_p, f_i)$ defines a complete intersection for $p+1 \leq i \leq m$. Then the principal value factors in*

$$T = \frac{1}{f_m} \cdots \frac{1}{f_{p+1}} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1}$$

commute with other principal value factors or residue factors, and the residue factors anticommute. In addition, if $p+1 \leq k \leq m$, we have

$$f_k T = \frac{1}{f_m} \cdots \widehat{\frac{1}{f_k}} \cdots \frac{1}{f_{p+1}} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1}, \tag{4.7}$$

and if $1 \leq j \leq p$, then

$$f_j T = 0. \tag{4.8}$$

Note that in case $f_i \in \tilde{\mathcal{O}}(Z)$, then the left-hand sides of (4.7) and (4.8) are defined by (4.6). Note that in the following lemmas, which we will use to prove Theorem 4.3, we assume that the functions are strongly holomorphic.

Lemma 4.4. *Assume that $f_1, f_2 \in \mathcal{O}(Z)$ and that $T \in \mathcal{PM}(Z)$ is of bidegree $(*, p)$. If $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} T \subseteq V$, for some analytic set $V \subseteq Z$ of codimension $\geq p+1$ in Z, then*

$$\frac{1}{f_1}\frac{1}{f_2} T = \frac{1}{f_2}\frac{1}{f_1} T. \tag{4.9}$$

If $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} T \subseteq V'$, for some analytic set V' of codimension $\geq p+2$ in Z, then

$$\frac{1}{f_1} \bar\partial \frac{1}{f_2} \wedge T = \bar\partial \frac{1}{f_2} \wedge \frac{1}{f_1} T, \tag{4.10}$$

and if in addition $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} \bar\partial T \subseteq V''$, for some analytic set V'' of codimension $\geq p+3$, then

$$\bar\partial \frac{1}{f_1} \wedge \bar\partial \frac{1}{f_2} \wedge T = -\bar\partial \frac{1}{f_2} \wedge \bar\partial \frac{1}{f_1} \wedge T. \tag{4.11}$$

Proof. Outside of Z_{f_1}, we have by Proposition 3.1 that $(1/f_1)(1/f_2)T = (1/f_2)(1/f_1)T$, since both are just multiplication of $(1/f_2)T$ with the smooth function $(1/f_1)$, and similarly outside of Z_{f_2}. Thus, we get that $(1/f_1)(1/f_2)T - (1/f_2)(1/f_1)T$ is a pseudomeromorphic current on Z of bidegree $(*, p)$ with support on $Z_{f_1} \cap Z_{f_2} \cap V$, which has codimension $\geq p+1$, so (4.9) follows by Proposition 3.2. Similarly outside of Z_{f_1}, we get $(1/f_1)\bar\partial(1/f_2) \wedge T = \bar\partial(1/f_2) \wedge (1/f_1)T$, so $(1/f_1)\bar\partial(1/f_2) \wedge T - \bar\partial(1/f_2) \wedge (1/f_1)T$ is a pseudomeromorphic current on Z of bidegree

$(*, p+1)$ and has support on $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} T$, so (4.10) follows by Proposition 3.2. For (4.11), we get by Theorem 4.2 and (4.10) that

$$\begin{aligned}
&\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}\wedge T = \bar\partial\left(\frac{1}{f_1}\bar\partial\frac{1}{f_2}\wedge T\right) + \frac{1}{f_1}\bar\partial\frac{1}{f_2}\wedge\bar\partial T\\
&= \bar\partial\left(\bar\partial\frac{1}{f_2}\wedge\frac{1}{f_1}T\right) + \frac{1}{f_1}\bar\partial\frac{1}{f_2}\wedge\bar\partial T\\
&= -\bar\partial\frac{1}{f_2}\wedge\bar\partial\frac{1}{f_1}\wedge T - \bar\partial\frac{1}{f_2}\wedge\frac{1}{f_1}\bar\partial T + \frac{1}{f_1}\bar\partial\frac{1}{f_2}\wedge\bar\partial T = -\bar\partial\frac{1}{f_2}\wedge\bar\partial\frac{1}{f_1}\wedge T
\end{aligned}$$

where the last equality holds because of (4.10) and the assumption of the support of $\bar\partial T$. □

Lemma 4.5. *Assume $f, g \in \mathcal{O}(Z)$, and $f/g \in \mathcal{O}(Z)$. If $T \in \mathcal{PM}(Z)$ has bidegree $(*, p)$ and $Z_g \cap \operatorname{supp} T \subseteq V$, for some analytic subset V of codimension $\geq p+1$, then*

$$f\left(\frac{1}{g}T\right) = \frac{f}{g}T.$$

Proof. Outside of Z_g, we can see $(1/g)T$ as multiplication by the smooth function $1/g$ by Proposition 3.1. Hence we have $f(1/g)T = (f/g)T$ since their difference is a pseudomeromorphic current with support on $Z_g \cap \operatorname{supp} T$, so it is 0 by Proposition 3.2. □

Proof of Theorem 4.3. First we observe that it is enough to prove the theorem in case f_i are strongly holomorphic, since if $\pi : Z' \to Z$ is the normalization of Z, and $f' = \pi^* f$, then f' is a complete intersection, and if the theorem holds in Z', it holds in Z by taking push-forward of the corresponding currents. Hence, we can assume that $f_i \in \mathcal{O}(Z)$, and the commutativity properties will then follow from Lemma 4.4. For example, if we want to see that $1/f_{i+1}$ and $1/f_i$ commute, we can apply Lemma 4.4 with

$$T = \frac{1}{f_{i-1}}\cdots\frac{1}{f_{p+1}}\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1},$$

and then multiply with $(1/f_m)\cdots(1/f_{i+2})$ from the left. In case some of the residue factors, say $f_{k+1}, \ldots, f_p$, are to the left of the principal value factors, then $Z_{(f_1,\ldots,f_k)}$ has codimension k in a neighborhood of $Z_f \supseteq \operatorname{supp} T$ and the result follows in the same way from Lemma 4.4. The other cases follow similarly from Lemma 4.4.

The equality (4.7) follows from Lemma 4.5 since Z_f has codimension p. By the first part of the theorem, we can assume that $j = p$ in (4.8). Then

$$\begin{aligned}
f_p\left(\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\right) &= \bar\partial\left(f_p\frac{1}{f_p}\wedge\bar\partial\frac{1}{f_{p-1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\right)\\
&= \bar\partial\left(\bar\partial\frac{1}{f_{p-1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\right) = 0
\end{aligned}$$

by (4.7), and Theorem 4.2. □

5. Multiplication of currents with weakly holomorphic functions

Now, we will return to the issue of multiplication of currents with weakly holomorphic functions. Assume $g \in \tilde{\mathcal{O}}(Z)$ and $S \in \mathcal{PM}(Z)$. Since $S \in \mathcal{PM}(Z)$, we have $S = \sum (\pi_\alpha)_* \tau_\alpha$, where τ_α are elementary currents on the complex manifolds Z_α. Given such a decomposition, since any normal modification of Z factors through the normalization, that is, $\pi_\alpha = \pi \circ \nu_\alpha$, for some $\nu_\alpha : Z_\alpha \to Z'$, we get a current S' in the normalization Z' of Z such that $\pi_* S' = S$ by taking the push-forward of τ_α to Z', i.e., $S' = \sum (\nu_\alpha)_* \tau_\alpha$. To define multiplication of the Coleff-Herrera product with the weakly holomorphic function g in (4.6), we defined it as the push-forward of $\pi^* g S'$. In general, the current S' will depend on the decomposition $S = \sum (\pi_\alpha)_* \tau_\alpha$. However, in (4.6) we had a canonical representative in the normalization, and hence the multiplication was well-defined. The following example however shows that this multiplication depends on this choice of representative.

Example 3. Let $\pi : \mathbb{C}^n \to \mathbb{C}^{2n}$ be defined by

$$\pi(t_1, \cdots, t_n) = (t_1, \cdots, t_{n-1}, t_1^2 t_n, \cdots, t_{n-1}^2 t_n, t_n^2, t_n^5).$$

Then π is proper and injective, so $\pi(\mathbb{C}^n) = Z$ is an analytic variety of dimension n. Since $(\partial \pi_j / \partial z_i)_{i,j}$ has full rank outside of $\{0\}$, $Z_{\text{sing}} \subseteq \{0\}$, and we will see below that actually $Z_{\text{sing}} = \{0\}$. Let

$$\tilde{S} = \bar{\partial}\frac{1}{t_1} \wedge \cdots \wedge \bar{\partial}\frac{1}{t_{n-1}} \wedge \bar{\partial}\frac{1}{t_n^3}$$

and $S = \pi_* \tilde{S}$. Then, since $d(t_n t_i^2) = t_i(2t_n dt_i + t_i dt_n)$ and $dt_n^5 = 5t_n^4 dt_n$, $dz_k \wedge S = 0$, where $k = n + i - 1$, for $i = 1, \ldots, n-1$ and $dz_{2n} \wedge S = 0$. Hence if $S.\xi \neq 0$, then ξ must be of the form $\xi = \xi_0 dz_1 \wedge \cdots \wedge dz_{n-1} \wedge dz_{2n-1}$. We have

$$S.\xi = \tilde{S}.\xi_0 dt_1 \wedge \cdots \wedge dt_{n-1} \wedge 2t_n dt_n =$$
$$2 \cdot (2\pi i)^n \left(\sum_{i=1}^{n-1} t_i^2 \frac{\partial}{\partial z_{n-1+i}} \xi_0 + 2t_n \frac{\partial}{\partial z_{2n-1}} \xi_0 + 5t_n^4 \frac{\partial}{\partial z_{2n}} \xi_0 \right) \Bigg|_{t=0} = 0,$$

and thus $S = 0$. However,

$$t_n \tilde{S}.\xi dt_1 \wedge \cdots \wedge dt_n^2 = 2(2\pi i)^n \xi(0)$$

so $\pi_*(t_n \tilde{S}) = \pi_*(\pi^* g \tilde{S}) \neq 0$, where $g \in \mathcal{O}_c(Z)$ is such that $\pi^* g = t_n$. Note that g is not strongly holomorphic at 0, and hence $Z_{\text{sing}} = \{0\}$.

Hence, the multiplication in (4.6) does not depend only on g and S, but also on the functions f defining S. Recall that the *pole set*, P_ϕ, of a meromorphic function ϕ is the set where ϕ is not strongly holomorphic. Recall also the definitions of the restriction operators $\mathbf{1}_V$ and $\mathbf{1}_{V^c}$ in

(3.2). If we require that the current we get in the multiplication has restriction 0 to P_ϕ, the multiplication is in fact uniquely defined in $\mathcal{PM}(Z)$, as the following proposition shows. This can in some cases be a natural condition, and in fact even automatic in some cases, see Corollary 5.2. However, in Example 3, since the common zero set of the functions defining S equals the pole set of g, we expect S and gS to have its support on P_g, and hence the condition is not very natural then.

Proposition 5.1. *Let $\mu \in \mathcal{PM}(Z)$ and $\phi \in \tilde{\mathcal{O}}(Z)$. Then, there exists a unique current, denoted $\phi\mu$, in $\mathcal{PM}(Z)$, such that $\phi\mu$ is just multiplication of the smooth function ϕ with the current μ outside of P_ϕ, and $\mathbf{1}_{P_\phi}(\phi\mu) = 0$. If $\mu = \pi_*\mu'$, where $\pi : Z' \to Z$ is the normalization of Z and $\mu' \in \mathcal{PM}(Z')$, then*

$$\phi\mu = \pi_*((\pi^*\phi)\mathbf{1}_{(\pi^{-1}(P_\phi))^c}\mu'). \tag{5.1}$$

Proof. First, we prove the uniqueness. Assume that T_1 and T_2 are two such currents, so that $T_1 - T_2$ has support on P_ϕ. Hence, $\mathbf{1}_{P_\phi^c}(T_1 - T_2) = 0$. But then,

$$T_1 - T_2 = \mathbf{1}_{P_\phi^c}(T_1 - T_2) + \mathbf{1}_{P_\phi}(T_1 - T_2) = 0,$$

since $\mathbf{1}_{P_\phi}T_1 = \mathbf{1}_{P_\phi}T_2 = 0$. Thus, we only need to prove that $\phi\mu$ in (5.1) satisfies the conditions in the proposition. It is clear that the right-hand side of (5.1) is just multiplication of ϕ with μ outside of P_ϕ. Hence, it remains to prove that $\mathbf{1}_{P_\phi}(\phi\mu) = 0$. However,

$$\mathbf{1}_{P_\phi}(\phi\mu) = \pi_*(\mathbf{1}_{\pi^{-1}(P_\phi)}(\pi^*\phi)\mathbf{1}_{(\pi^{-1}(P_\phi))^c}\mu') = 0,$$

since $\mathbf{1}_V\mathbf{1}_{V^c} = \mathbf{1}_V(1 - \mathbf{1}_V) = 0$ because $\mathbf{1}_V\mathbf{1}_V = \mathbf{1}_V$, and $\mathbf{1}_V$ commutes with multiplication with smooth functions. □

Corollary 5.2. *Assume that $\mu \in \mathcal{PM}(Z)$ is of bidegree $(*, p)$ and $\phi \in \tilde{\mathcal{O}}(Z)$ is such that P_ϕ has codimension $\geq p + 1$ in Z. Then there exists a unique current $\phi\mu \in \mathcal{PM}(Z)$ such that $\phi\mu$ coincides with the usual multiplication of ϕ with μ outside of P_ϕ. If $\mu = \pi_*\mu'$, where $\pi : Z' \to Z$ is the normalization of Z and $\mu' \in \mathcal{PM}(Z')$, then*

$$\phi\mu = \pi_*((\pi^*\phi)\mu'). \tag{5.2}$$

Proof. By Proposition 5.1, the only thing we need to prove is that for any $T \in \mathcal{PM}(Z)$ and $T' \in \mathcal{PM}(Z')$ of bidegree $(*, p)$, we have $\mathbf{1}_{P_\phi}T = 0$ and $\mathbf{1}_{\pi^{-1}(P_\phi)}T' = 0$. However, since P_ϕ has codimension $\geq p + 1$, $\pi^{-1}(P_\phi)$ has codimension $\geq p + 1$ by Lemma 2.2. Hence, $\mathbf{1}_{P_\phi}T = 0$ and $\mathbf{1}_{\pi^{-1}(P_\phi)}T' = 0$ by Proposition 3.2, since the currents have support on P_ϕ and $\pi^{-1}(P_\phi)$ respectively. □

Note, in particular that if Z_{sing} has codimension $\geq p + 1$, the condition of the codimension of P_g is automatically satisfied for any weakly holomorphic function $g \in \tilde{\mathcal{O}}(Z)$.

Another question is whether the Coleff-Herrera product could be defined as the analytic continuation of an integral on Z rather than Z'. A natural way to do this would be to try to regularize in (4.3) by factors $\bar{\partial}|F_i|^{2\lambda_i}$ instead of $\bar{\partial}|f_i|^{2\lambda_i}$, where F_i is a tuple of strongly holomorphic functions such that $Z_{F_i} = P_{1/f_i}$. However, the analytic continuation to $\lambda = 0$ will in general not coincide with our definition, even if f defines a complete intersection, as the following example shows.

Example 4. Let $Z = \{z \in \mathbb{C}^3 \mid z_1^3 = z_2^2\} = V \times \mathbb{C}$, which has normalization $\pi(s,t) = (s^2, s^3, t)$, and let $\pi^* f_1 = s^2$ and $\pi^* f_2 = (1+s)t$. Then $Z_f = \{0\}$, so f is a complete intersection. Note that $\pi^*(1/f_2) = (1/t)(1 - s + O(s^2))$ for $|s| < 1$, and that holomorphic functions in s at the origin correspond to strongly holomorphic functions on V at the origin precisely when the Taylor expansion at the origin contains no term s. Thus $P_{1/f_2} = \pi(\{s = 0\} \cup \{s = -1\} \cup \{t = 0\})$, so if $\{F = 0\} \supseteq P_{1/f_2}$, then $\{F = 0\} \supseteq Z_{f_1}$. Thus $(\bar{\partial}|F|^{2\lambda}/f_2) \wedge \bar{\partial}(1/f_1) = 0$ for $\operatorname{Re} \lambda \gg 0$. However, we have

$$\bar{\partial}\frac{1}{f_2} \wedge \bar{\partial}\frac{1}{f_1}.\varphi dz_1 \wedge dz_3 = \frac{1}{1+s}\bar{\partial}\frac{1}{t} \wedge \bar{\partial}\frac{1}{s^2}.\varphi(s^2, s^3, t) ds^2 \wedge dt = 4\pi i \varphi(0),$$

so $\bar{\partial}(1/f_1) \wedge \bar{\partial}(1/f_2)$ is non-zero.

6. Bochner-Martinelli type residue currents

We will show that we can define a Bochner-Martinelli type residue current associated with a tuple of weakly holomorphic functions, either by using a similar approach as for the Coleff-Herrera product with the help of the normalization, or by defining it intrinsically on Z by means of analytic continuation. In view of Example 4, it is not clear how to do this directly for the Coleff-Herrera product. In addition, we will show that for weakly holomorphic functions defining a complete intersection, the Coleff-Herrera product and the Bochner-Martinelli current coincide, Theorem 6.3.

Let $f = (f_1, \dots, f_p)$ be weakly holomorphic. We will follow the approach by Andersson from [1], and make the identification $f = \sum f_i e_i^*$, where $(e_1, \cdots, e_p)$ is a frame for a trivial vector bundle E over Z, and $(e_1^*, \dots, e_p^*)$ is the dual frame. Since we will only use the case of trivial vector bundles, this identification merely serves as a notational convenience. Then, on the set where f is strongly holomorphic, $\nabla_f := \delta_f - \bar{\partial}$ induces a complex on currents on Z with values in $\bigwedge E$, where δ_f is interior multiplication with f. To construct the Bochner-Martinelli current we define

$$\sigma = \sum \frac{\bar{f}_i e_i}{|f|^2} \quad \text{and} \quad u = \sum_{k=0}^{p-1} \sigma \wedge (\bar{\partial}\sigma)^k. \tag{6.1}$$

Note that outside of $Z_f \cup P_{f_1} \cup \cdots \cup P_{f_p}$, both u and σ are smooth, and $\nabla_f u = 1$.

Recall that a *universal denominator* at a germ (Z, z) is a strongly holomorphic function h, not vanishing on any irreducible component of (Z, z) such that $h\tilde{\mathcal{O}}_{Z,z} \subseteq \mathcal{O}_{Z,z}$. For each $z \in Z$, there always exist a universal denominator h, such that h is a universal denominator in a neighborhood of z, see for example [15], Theorem Q.2.

Proposition 6.1. *Assume that $f = (f_1, \cdots, f_p)$ is weakly holomorphic on Z. Let F be a tuple of strongly holomorphic functions, such that $\{F = 0\} \supseteq Z_f$, and $\{F = 0\}$ does not contain any irreducible component of Z, and let h be a universal denominator on Z. Then the forms $|hF|^{2\lambda} u$ and $\bar\partial |hF|^{2\lambda} \wedge u$ are arbitrarily smooth if* $\operatorname{Re}\lambda \gg 0$*, and have current-valued analytic continuations to* $\operatorname{Re}\lambda > -\epsilon$ *for some* $\epsilon > 0$*. The currents*

$$U^f := |hF|^{2\lambda} u|_{\lambda=0} \quad \text{and} \quad R^f := \bar\partial |hF|^{2\lambda} \wedge u|_{\lambda=0} \tag{6.2}$$

are independent of the choice of F and h, and if $\pi : Y \to Z$ is a modification of Z, then $U^f = \pi_* U^{\pi^* f}$ and $R^f = \pi_* R^{\pi^* f}$.

Proof. We first show that $|hF|^{2\lambda} u$ and $\bar\partial |hF|^{2\lambda} \wedge u$ are arbitrarily smooth when $\operatorname{Re}\lambda \gg 0$. Since $\bar\partial |hF|^{2\lambda} = |hF|^{2(\lambda-1)} \bar\partial |hF|^2$, it is enough to prove this for $|hF|^{2\lambda} u$. We let $g_i := h f_i$, where $g_i \in \mathcal{O}(Z)$ since h is a universal denominator. If we differentiate u outside of $\{h = 0\} \cup Z_f$, we get terms of the form $\xi/(h^k |f|^{2n})$, where ξ is smooth, since if $f_i = g_i/h$, the terms in u are smooth except for factors h and $|f|^2$ in the denominators. Thus, we only need to see that $|hF|^{2\lambda}/(h^k |f|^{2n})$ tends to 0 on $\{h = 0\} \cup Z_f$. This is clear outside of Z_f if $\operatorname{Re}\lambda \gg 0$, so we need to prove that $|hF|^{2\lambda}/|f|^{2n}$ tends to 0 on Z_f. If we multiply the numerator and denominator by $|h|^{2n}$, we get

$$|h|^{2n} |hF|^{2\lambda} / (|hf|^{2n}). \tag{6.3}$$

We note that hf is strongly holomorphic, and in fact, $\{hF = 0\} \supseteq \{hf = 0\}$ because

$$Z_{hf} = \pi(Z_{\pi^*(hf)}) = \pi(Z_{\pi^* h}) \cup \pi(Z_{\pi^* f}) = Z_h \cup \pi(Z_{\pi^* f}) = \{h = 0\} \cup Z_f,$$

by Proposition 2.3 and the fact that π is surjective. Thus, (6.3) will tend to 0 on Z_f by the Nullstellensatz if $\operatorname{Re}\lambda \gg 0$.

Now, we assume that Z is smooth. Then we can take $F = f$ and $h \equiv 1$, and in that case, the proposition is the existence part of Theorem 1.1 in [1], except for the fact that $U^f = \pi_* U^{\pi^* f}$ and $R^f = \pi_* R^{\pi^* f}$, which however easily follows by analytic continuation. To see that the definition of R^f is independent of the choice of F, we see from the proof of Theorem 1.1 in [1] that $\bar\partial |F|^{2\lambda} \wedge u$ acting on a test form φ becomes, with a suitable resolution of singularities $\pi : \tilde X \to X$, a finite sum of terms of the kind

$$\int \frac{\bar\partial |u\mu_1|^{2\lambda}}{\mu_2} \wedge \sigma' \wedge \pi^*\varphi, \tag{6.4}$$

where μ_1 and μ_2 are monomials such that $\{\mu_1 = 0\} \supseteq \{\mu_2 = 0\}$, u is non-zero and σ' is smooth. Thus, it is enough to observe that the value at $\lambda = 0$ of (6.4) is independent of μ_1 (where $u\mu_1$ is the pull-back of F), as long as $\{\mu_1 = 0\} \supseteq \{\mu_2 = 0\}$. In the same way, one sees that the definition of U^f is independent of the choice of F.

Now, if f is weakly holomorphic, and $\pi : \tilde Z \to Z$ is a resolution of singularities, from the smooth case we know that $\bar\partial|\pi^*(hF)|^{2\lambda} \wedge \pi^* u$ has a current-valued analytic continuation to $\lambda = 0$ independent of the choice of hF. Hence, the weakly holomorphic case follows by taking push-forward, since $\bar\partial|hF|^{2\lambda} \wedge u = \pi_*(\bar\partial|\pi^*(hF)|^{2\lambda} \wedge \pi^* u)$ for $\operatorname{Re}\lambda \gg 0$. $\square$

In fact, to prove the existence of U^f and R^f, defined by (6.2), it is sufficient to use $|F|^{2\lambda}u$ and $\bar\partial|F|^{2\lambda} \wedge u$, which can be seen are integrable on Z if $\operatorname{Re}\lambda \gg 0$ by going back to the normalization. However, the addition of the universal denominator h ensures that the forms are (arbitrarily) smooth if $\operatorname{Re}\lambda \gg 0$.

The following properties of the Bochner-Martinelli current, R^f, are well-known in the smooth case, see [22] and [1].

Proposition 6.2. *Let $f = (f_1, \cdots, f_p)$ be weakly holomorphic, and assume that $p' = \operatorname{codim} Z_f$. The current R^f has support on $V = Z_f$, and there is a decomposition $R^f = \sum_{k=p'}^p R_k$, where $R_k \in \mathcal{PM}(Z)$ is a $(0,k)$-current with values in $\bigwedge^k E$. In addition, if f is strongly holomorphic, then $R^f = 1 - \nabla_f U^f$.*

Proof. In case Z is a complex manifold, this is parts of Theorem 1.1 in [1], except for the fact that $R_k \in \mathcal{PM}(Z)$. However, that R_k is pseudomeromorphic can, as was noted in [4], easily be seen from the proof of Theorem 1.1 in [1]. The proposition then follows in case of an analytic space, by taking push-forward from a resolution of singularities, except for the fact that $R^f = \sum_{k=p'}^p R_k$, where $p' = \operatorname{codim} Z_f$, since modifications does not in general preserve codimensions of subvarieties. However, we get that $R^f = \sum_{k=0}^p R_k$, where $R_k \in \mathcal{PM}(Z)$ is a $(0,k)$-current, and R_k has support on Z_f. Thus, by Proposition 3.2, $R_k = 0$ for $k < \operatorname{codim} Z_f = p'$. $\square$

Remark 5. If the mapping f is weakly holomorphic, as we saw in Example 3, we do not have a well-defined multiplication of weakly holomorphic functions with pseudomeromorphic currents on Z. Hence, the formula $R^f = 1 - \nabla_f U^f$ in the strongly holomorphic case does not necessarily have any meaning if f is weakly holomorphic. However, one can give this multiplication meaning by Proposition 5.1. With this definition of multiplication, one can verify that

$$R^f = 1 - \nabla_f U^f,$$

if f is weakly holomorphic. This can be seen by using that this formula holds in the normalization, together with the fact that $U^{f'}$ has the standard extension property, SEP, i.e., that $\mathbf{1}_{\{h=0\}} U^f = 0$ for any tuple h of strongly holomorphic functions not vanishing on any irreducible component of Z. This follows from that $U^{f'}$ is a principal value current, i.e., when $U^{f'}$ is written as a sum of push-forwards of elementary currents, the elementary currents contain no residue factors, and hence have the SEP.

Theorem 6.3. *If $f = (f_1, \dots, f_p)$ is weakly holomorphic forming a complete intersection, then*

$$R^f = \bar\partial \frac{1}{f_p} \wedge \ldots \wedge \bar\partial \frac{1}{f_1} \wedge e_1 \wedge \cdots \wedge e_p.$$

Proof. To begin with, we will assume that f is strongly holomorphic. The proof will follow the same idea as the proof in the smooth case in [2], Theorem 3.1. Let

$$V = \frac{1}{f_1} e_1 + \frac{1}{f_2} \bar\partial \frac{1}{f_1} \wedge e_1 \wedge e_2 + \cdots + \frac{1}{f_p} \bar\partial \frac{1}{f_{p-1}} \wedge \cdots \wedge \frac{1}{f_1} \wedge e_1 \wedge \cdots \wedge e_p.$$

Then, by Proposition 4.3, V satisfies

$$\nabla_f V = 1 - \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \wedge e,$$

where $e = e_1 \wedge \cdots \wedge e_p$. Following the proof of Theorem 3.1 in [2], locally, assume $Z \subseteq \Omega \subseteq \mathbb{C}^n$, ω is an arbitrary neighborhood of Z_f in Ω and χ is a smooth function with support on ω which is $\equiv 1$ in a neighborhood of Z_f. By Proposition 6.2, since f defines a complete intersection, $R^f = R^f_p$, and we denote the component $R^f_p = \mu \wedge e$. Let $i : Z \to \Omega$ be the inclusion, and let $g = i^*\chi - i^*(\bar\partial\chi) \wedge u$. Then, since $\nabla_f u = 1$ on $\operatorname{supp} \bar\partial\chi$, $\nabla_f g = 0$, and hence

(6.5)

$$\nabla_f(g \wedge (U^f - V)) = g \wedge \nabla_f(U^f - V) = g_0(\mu^f - \mu) \wedge e = (\mu^f - \mu) \wedge e,$$

where $g_0 = \chi$ is the component of bidegree $(0,0)$ in g, which is 1 in a neighborhood of $\operatorname{supp}(\mu^f - \mu)$. A current T is said to have the standard extension property, SEP, with respect to an analytic variety W if for any holomorphic function h such that h is not identically 0 on any irreducible component of W, then $|h|^{2\lambda} T|_{\lambda=0} = T$. Since μ and μ^f are currents in $\mathcal{PM}(Z)$ of bidegree $(0,p)$, with support on $W = \{f = 0\}$, μ and μ^f have the SEP, since if h does not vanish on any irreducible component of W, $\mu - |h|^{2\lambda}\mu|_{\lambda=0}$ has support on $W \cap \{h = 0\}$, which has codimension $\geq p+1$, and by Proposition 3.2 it is 0. Also, μ and μ^f are $\bar\partial$-closed and are annihilated by $\bar I_W$, see Proposition 3.2, so

$i_*\mu, i_*\mu^f \in CH_W$, where CH_W denotes $\bar\partial$-closed $(0, \operatorname{codim} W)$-currents with support on W satisfying the SEP. By Lemma 3.3 in [2], we know that a $\bar\partial$-closed current in CH_W cannot be equal to $\bar\partial\nu$, where ν can be chosen with support arbitrarily close to W, unless it is 0. Hence, by looking at the components of top degree in (6.5), we have $i_*(\mu - \mu^f) = 0$, so $\mu = \mu^f$.

Now, if f_i are weakly holomorphic, then the current R^f will be the push-forward of the corresponding current $R^{\pi^* f}$, where $\pi : Z' \to Z$ is the normalization of Z, and the same holds for the Coleff-Herrera product μ^f. Hence, equality holds in the normalization, and taking push-forward we get equality in the general case. $\square$

7. The transformation law

With the Bochner-Martinelli type currents developed in the previous section, we will now prove the transformation law for Coleff-Herrera products of weakly holomorphic functions.

Theorem 7.1. *Assume that $f = (f_1, \cdots, f_p)$ and $g = (g_1, \cdots, g_p)$ are weakly holomorphic, defining complete intersections, and that there exists a matrix A of weakly holomorphic functions such that $g = Af$. Then*

$$\bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} = (\det A)\bar\partial\frac{1}{g_p} \wedge \cdots \wedge \bar\partial\frac{1}{g_1}.$$

If A is invertible, one can prove the transformation law with the help of Theorem 6.3 together with the fact that the Bochner-Martinelli current is independent of the metric chosen to define σ^f (here, in (6.1), σ^f is defined with respect to the trivial metric on E), see [1]. We will see that we can use a similar idea even in the case that A is not invertible. In [13] Denkowski proved the transformation law for c-holomorphic functions based on a more direct approach.

To begin with, we assume that f, g and A are strongly holomorphic. As in the previous section, we will identify f and g with sections of vector bundles, however we will here identify them with sections of two different vector bundles. Let E and E' be trivial holomorphic vector bundles over Z with frames e and e', and make the identifications $f = \sum f_i e_i^*$, $g = \sum g_i e_i'^*$ and $A \in \operatorname{Hom}(E', E)$ such that $g = fA$.

Lemma 7.2. *Let $\bigwedge A : \bigwedge E' \to \bigwedge E$ denote the linear extension of the mapping $(\bigwedge A)(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$. Then $\delta_f(\bigwedge A) = (\bigwedge A)\delta_g$.*

Proof. Note first that $\delta_f A e_j' = g_j = \delta_g e_j'$. Hence, we have

$$\delta_f(\bigwedge A)(e_{i_1}' \wedge \cdots \wedge e_{i_k}') = \delta_f(Ae_{i_1}' \wedge \cdots \wedge Ae_{i_k}')$$
$$= \sum (-1)^{j-1} Ae_{i_1}' \wedge \cdots \wedge \delta_f(Ae_{i_j}') \wedge \cdots \wedge Ae_{i_k}'$$
$$= \sum (-1)^{j-1} (\bigwedge A)(e_{i_1}' \wedge \cdots \wedge \delta_g e_{i_j}' \wedge \cdots \wedge e_{i_k}') = (\bigwedge A)\delta_g(e_{i_1}' \wedge \cdots \wedge e_{i_k}').$$

□

To relate the currents μ^f and μ^g, we will first derive a relation between the currents U^f and U^g as defined by (6.2).

Lemma 7.3. *If f and g are strongly holomorphic and defining complete intersections, then there exists a current R_1 such that $U^f - (\bigwedge A)U^g = \nabla_f R_1$.*

Proof. Let σ, u, σ' and u' be the forms defined by (6.1) corresponding to f and g. Since A is holomorphic, $(\bigwedge A)\bar\partial\sigma' = \bar\partial(A\sigma')$ outside of $\{g = 0\}$, and hence if we let $u'_A = \sum(A\sigma') \wedge (\bar\partial A\sigma')^{k-1}$, then $\nabla_f u'_A = 1$ outside of $\{g = 0\}$ by Lemma 7.2. Thus, if $\operatorname{Re}\lambda \gg 0$,

$$\nabla_f(|g|^{2\lambda}u'_A \wedge u) = |g|^{2\lambda}u - |g|^{2\lambda}u'_A - \bar\partial|g|^{2\lambda} \wedge u'_A \wedge u. \tag{7.1}$$

We want to see that all the terms in (7.1) have current-valued analytic continuations to $\lambda = 0$. First, we note that since $\{g = 0\} \supseteq \{f = 0\}$, $|g|^{2\lambda}u|_{\lambda=0} = U^f$ by Proposition 6.1, and since $u'_A = (\bigwedge A)u'$ we get that $|g|^{2\lambda}u'_A|_{\lambda=0} = (\bigwedge A)U^g$. Thus it remains to see that the left-hand side of (7.1) has an analytic continuation to $\lambda = 0$, and that the analytic continuation of the last term vanishes at $\lambda = 0$. To see that those terms have analytic continuations to $\lambda = 0$ is similar to showing the existence of the Bochner-Martinelli currents U^f and R^f. If we recall briefly the proof of the existence of U^f and R^f in [1], the key step was that $\sigma \wedge (\bar\partial\sigma)^{k-1}$ is homogeneous with respect to f in the sense that if $f = f_0 f'$, then $\sigma \wedge (\bar\partial\sigma)^{k-1} = (1/f_0^k)\sigma_0 \wedge (\bar\partial\sigma_0)^{k-1}$, where σ_0 is smooth if $|f'| \neq 0$. By blowing up along the ideals $(f_1, \dots, f_p)$ and $(g_1, \dots, g_p)$ followed by a resolution of singularities, see [5], we can assume that locally $\pi^* f = f_0 h$ and $\pi^* g = g_0 g'$, where $h \neq 0$, $g' \neq 0$, and by a further resolution of singularities, we can assume that locally f_0, g_0 are monomials. Since $\{g = 0\} \supseteq \{f = 0\}$, we get that $\{g_0 = 0\} \supseteq \{f_0 = 0\}$. Thus, by the homogeneity of $\sigma' \wedge (\bar\partial\sigma')^{k-1}$ and $\sigma \wedge (\bar\partial\sigma)^{l-1}$ with respect to f and g, we get, since $u'_A = (\bigwedge A)u'$, that $|g|^{2\lambda}u'_A \wedge u$ and $\bar\partial|g|^{2\lambda} \wedge u'_A \wedge u$ acting on a test form φ becomes finite sums of the form

$$\int \frac{|v|^{2\lambda}|g_0|^{2\lambda}}{(g_0)^k f_0^l}\xi_{k,l} \wedge \pi^*\varphi \quad \text{and} \quad \int \frac{\bar\partial(|v|^{2\lambda}|g_0|^{2\lambda})}{(g_0)^k f_0^l} \wedge \xi_{k,l} \wedge \pi^*\varphi,$$

where $\xi_{k,l}$ are smooth $(0, k + l - 2)$-forms. Thus both have analytic continuations to $\lambda = 0$, and $R_2 := \bar\partial|g|^{2\lambda} \wedge u'_A \wedge u|_{\lambda=0}$ has support on $\{g = 0\}$. Since $R_2 \in \mathcal{PM}(Z)$ and consists of terms of bidegree $(0, k + l - 1)$, where $k + l \leq p$, with support on $\{g = 0\}$ which has codimension p, we get that $R_2 = 0$ by Proposition 3.2. Thus, if we let $R_1 := |g|^{2\lambda}u'_A \wedge u|_{\lambda=0}$, we get that $\nabla_f R_1 = U^f - (\bigwedge A)U^g$. □

Now we are ready to prove the transformation law.

Proof of Theorem 7.1. Assume first that f, g and A are strongly holomorphic, and make the same identifications as after the statement of Theorem 7.1. Since $(\bigwedge A)R^g = (\bigwedge A)(1 - \nabla_g U^g) = 1 - \nabla_f (\bigwedge A)U^g$ by Lemma 7.2, we get from Lemma 7.3 that

$$(\bigwedge A)R^g - R^f = \nabla_f \left((\bigwedge A)U^g - U^f \right) = \nabla_f^2 R_1 = 0,$$

so

$$(\bigwedge A)R^g = R^f.$$

Thus, we get by Theorem 6.3 that

$$(\bigwedge A) \left(\mu^g \wedge e_1' \wedge \cdots \wedge e_p' \right) = \mu^f \wedge e_1 \wedge \cdots \wedge e_p,$$

and since the left-hand side is equal to

$$(\det A)\mu^g \wedge e_1 \wedge \cdots \wedge e_p,$$

the transformation law follows. Now, if f, g and A are weakly holomorphic, the transformation law follows since equality must hold in the normalization because the pullback of f and g define complete intersections in the normalization. Hence, equality must hold also in Z by taking push-forward. $\square$

8. The Poincaré-Lelong formula

Let $f_1, \cdots, f_p$ be strongly holomorphic functions forming a complete intersection. The Poincaré-Lelong formula says that

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \wedge df_1 \wedge \cdots \wedge df_p = [Z_f] = \sum \alpha_i [V_i], \tag{8.1}$$

where V_i are the irreducible components of Z_f and $[Z_f]$ is the integration current on Z_f with multiplicities. In case $p = \dim Z$ the multiplicity α_i at a point $x_i \in Z_f$ is given as the number of elements near x_i of a generic fiber of f. In case $p < \dim Z$ the multiplicity is given as the intersection multiplicity of Z_f with L, where L is a plane of dimension $\dim Z - p$ transversal to Z_f. For a thorough discussion of the multiplicities see [11], and for a proof of the Poincaré-Lelong formula see Section 3.6 in [12].

Now, if f_i are weakly holomorphic functions defining a complete intersection, we can give a relatively short proof that a formula similar to (8.1) holds in Z. In the strongly holomorphic case, assuming $Z \subseteq \Omega \subseteq \mathbb{C}^n$, $i_*[Z_f]$ can be seen either as the intersection of the holomorphic chains Z_{F_i} with Z, where F_i are some holomorphic extensions of f_i to Ω, or as a product of closed positive currents, see [11], that is

$$i_*[Z_f] = [Z_{F_1} \cdot \cdots \cdot Z_{F_p} \cdot Z] = [Z_{F_1}] \wedge \cdots \wedge [Z_{F_p}] \wedge [Z].$$

However, these types of products are in general only defined in case $Z_{F_1} \cap \cdots \cap Z_{F_p} \cap Z$ has codimension equal to $\operatorname{codim} Z + \sum \operatorname{codim} Z_{F_i}$. Since zero sets of weakly holomorphic functions are in general not zero sets of strongly holomorphic functions, as we saw in Example 1, we cannot expect to have a similar interpretation for weakly holomorphic functions, since there are no natural counterparts to the holomorphic $(n-1)$-chains Z_{F_i} or closed positive $(1,1)$-currents $[Z_{F_i}]$.

From now on, we assume that $f = (f_1, \ldots, f_p)$ is weakly holomorphic defining a complete intersection. Let $\pi : Z' \to Z$ be the normalization of Z, so that in particular, π is a finite proper holomorphic map. Since $f' = \pi^* f$ forms a complete intersection, (8.1) holds for f' in the normalization. Note that, if V_i are the irreducible components of $Z_{f'}$, then $W_i := \pi(V_i)$ are irreducible in Z. If $f : V \to W$ is a branched holomorphic cover with exceptional set E, we say that f is a **-covering* if $W \setminus E$ is a connected manifold. In particular, this means that the sheet-number of f is constant outside the exceptional set. By the Andreotti-Stoll theorem, see [17], if $f : V \to W$ is a finite proper holomorphic map, V has constant dimension and W is irreducible, then f is a *-covering. If $V \subset Z'$ is an irreducible component of $Z_{f'}$ and we consider $\pi|_V : V \to W$, where $W = \pi(V)$, it is a finite proper holomorphic map satisfying the conditions required for the Andreotti-Stoll theorem. Hence, there exists an integer k such that $\pi|_V$ is a k-sheeted finite branched holomorphic covering. Thus $\pi_*\alpha[V] = k\alpha[W]$. For $f = (f_1, \cdots, f_p)$ a weakly holomorphic mapping forming a complete intersection, we define the left-hand side of (8.1) as the push-forward of the corresponding current in the normalization. Thus, since we have by (8.1) that

$$\frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} \wedge df_1 \wedge \cdots \wedge df_p = \pi_*[Z_{f'}],$$

we have proved the following.

Theorem 8.1. *Let $f = (f_1, \cdots, f_p)$ be a weakly holomorphic mapping forming a complete intersection. Then*

$$\frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} \wedge df_1 \wedge \cdots \wedge df_p = \sum \beta_i[W_i] \tag{8.2}$$

where $\beta_i \in \mathbb{N}$ and W_i are the irreducible components of $W = Z_f$. More explicitly, if $[Z_{f'}] = \sum \alpha_i[V_i]$ and say $V_{i_1}, \cdots, V_{i_k}$ are the sets V_j such that $\pi(V_j) = W_i$, then $\beta_i = \sum k_{i_j}\alpha_{i_j}$, where k_j is the number of elements in a generic fiber of $\pi|_{V_j}$.

Remark 6. In [13] Denkowski proves the Poincaré-Lelong formula for $f = (f_1, \ldots, f_p) \in \mathcal{O}_c{}^{\oplus p}(Z)$ (based on his construction on Γ_f, however as for the Coleff-Herrera product in Proposition 4.1 our definition coincides with his). In that case, it gives a different interpretation of the multiplicities as the intersection cycle

$$\frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} \wedge df_1 \wedge \cdots \wedge df_p = \pi_*([\Gamma_f] \cdot [Z \times \{0\}]),$$

where $\pi : Z \times \mathbb{C}^p \to Z$ is the projection.

Note that if f is weakly holomorphic, since f is in general not smooth on Z_{sing}, df is not in general defined on all Z (although its pullback to the normalization has a smooth extension to all of Z') so, as for multiplication with weakly holomorphic functions in Example 3, it might for example happen that $\bar\partial(1/f) = 0$ while $\bar\partial(1/f) \wedge df \neq 0$. For example, if $Z = \{z^3 = w^2\}$, $\pi(t) = (t^2, t^3)$ and $f = w/z \in \tilde{\mathcal{O}}(Z)$, that is $\pi^* f = t$, then $\bar\partial(1/f) = 0$ while $\bar\partial(1/f) \wedge df = 2\pi i[0]$, as expected, since $Z_f = \{0\}$.

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ON THE DUALITY THEOREM ON AN ANALYTIC VARIETY

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Abstract. The duality theorem for Coleff-Herrera products on a complex manifold says that if $f = (f_1, \dots, f_p)$ defines a complete intersection, then the annihilator of the Coleff-Herrera product μ^f equals (locally) the ideal generated by f. This does not hold unrestrictedly on an analytic variety Z. We give necessary, and in many cases sufficient conditions for when the duality theorem holds. These conditions are related to how the zero set of f intersects certain singularity subvarieties of the sheaf $\mathcal{O}_Z$.

1. Introduction

Let $f = (f_1, \dots, f_p)$ be a tuple of holomorphic functions on an analytic variety Z, where we throughout the article will assume that Z has pure dimension. The *Coleff-Herrera product* of f, as introduced in [9], can be defined by

$$\bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} . \varphi := \int_Z \frac{\bar\partial |f_p|^{2\lambda} \wedge \dots \bar\partial |f_1|^{2\lambda}}{f_p \cdots f_1} \wedge \varphi \Big|_{\lambda=0} . \tag{1.1}$$

Here, φ is a test form, and the integral on the right-hand side is analytic in λ for $\operatorname{Re} \lambda \gg 0$, and has an analytic continuation to $\lambda = 0$, and $|_{\lambda=0}$ denotes this value. We will also denote the Coleff-Herrera product of f by μ^f. The definition (1.1) is different from the original one, but in the case we focus on here, that f defines a *complete intersection*, i.e., that $\operatorname{codim} Z_f = p$, various different definitions including this definition and the original definition by Coleff and Herrera coincide, also on a singular variety, see [16].

If f defines a complete intersection, the *duality theorem*, proven by Dickenstein and Sessa, [11], and Passare, [18], gives a close relation between the Coleff-Herrera product of f and the ideal $\mathcal{J}(f_1, \dots, f_p)$ generated by f. This is done by means of the annihilator, $\operatorname{ann} \mu^f$, of μ^f, i.e., the holomorphic functions g such that $g\mu^f = 0$.

Theorem 1.1. *Let $f = (f_1, \dots, f_p)$ be a holomorphic mapping on a complex manifold defining a complete intersection. Then locally, $\mathcal{J}(f_1, \dots, f_p) = \operatorname{ann} \mu^f$.*

The Coleff-Herrera product of a holomorphic mapping is a current on Z. Currents on singular varieties can be defined in a similar way as on manifolds, i.e., as linear functionals on test-forms, see for example [15].

However, currents on Z also has a characterization in terms of currents in the ambient space: If $i : Z \to \Omega$ is the inclusion, $\operatorname{codim} Z = k$, and μ is a (p,q)-current on Z, then $i_*\mu$ is a $(k+p,k+q)$-current on Ω that vanishes on all forms that vanish on Z. Conversely, if T is a $(k+p,k+q)$-current on Ω, that vanishes on all forms that vanish on Z, then T defines a unique (p,q)-current T' on Z such that $i_*T' = T$. When we consider the Coleff-Herrera product in the ambient space, i.e., $i_*\mu^f$, we will denote it by

$$\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge[Z],$$

and in fact, by analytic continuation, it can be defined by

$$\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge[Z] = \left.\frac{\bar\partial|f_p|^{2\lambda}\wedge\cdots\wedge\bar\partial|f_1|^{2\lambda}}{f_p\cdots f_1}\wedge[Z]\right|_{\lambda=0}.$$

On an analytic variety, one can find rather simple examples of functions annihilating the Coleff-Herrera product of a complete intersection without lying in the ideal. However, we have an inclusion in one of the directions, see [9], Theorem 1.7.7.

Theorem 1.2. *If $f = (f_1,\dots,f_p)$ are holomorphic on Z, defining a complete intersection, then $\mathcal{J}(f_1,\dots,f_p) \subseteq \operatorname{ann}\mu^f$.*

In this article, we discuss this inclusion, and give conditions for when the inclusion is an equality, and when the inclusion is strict.

Throughout this article, we will only discuss the duality theorem for strongly holomorphic functions on Z, i.e., functions f on Z, which are locally the restriction of holomorphic functions in the ambient space, denoted $f \in \mathcal{O}(Z)$. When we say holomorphic functions, we refer to strongly holomorphic functions. However, we will sometimes refer to them as strongly holomorphic functions, to make a distinction to weakly holomorphic, which we use in the introduction to provide examples. Recall that a function $f : Z_{\text{reg}} \to \mathbb{C}$ is *weakly holomorphic* on Z, denoted $f \in \tilde{\mathcal{O}}(Z)$, if f is holomorphic on Z_{reg}, and f is locally bounded at Z_{sing}. Recall also that a germ of a variety, (Z,z), is said to be *normal* if $\mathcal{O}_{Z,z} = \tilde{\mathcal{O}}_{Z,z}$, and that the *normalization* of a variety Z is the unique (up to analytic isomorphism) variety Z' together with a finite proper surjective holomorphic map $\pi : Z' \to Z$ such that $\pi|_{Z'\setminus\pi^{-1}(Z_{\text{sing}})} : Z' \setminus \pi^{-1}(Z_{\text{sing}}) \to Z_{\text{reg}}$ is a biholomorphism.

One of the reasons we do not have equality in Theorem 1.2 is because of weakly holomorphic functions, namely if $f = (f_1,\dots,f_p)$ is strongly holomorphic and defining a complete intersection, and $g = \sum a_i f_i$ is strongly holomorphic while the functions a_i are only weakly holomorphic, then by Theorem 4.3 in [15], $g\mu^f = 0$, but it might very well happen that the a_i cannot be chosen to be strongly holomorphic. For example, let $Z = \{z^3 = w^2\} \subseteq \mathbb{C}^2$, which has normalization $\pi(t) = (t^2,t^3)$, and let $f \in \tilde{\mathcal{O}}(Z)$ be such that $\pi^* f = t$. Then $f^2 = z$ and $f^3 = w$ on Z, so that $f^2, f^3 \in \mathcal{O}(Z)$ and $f^3\bar\partial(1/f^2) = 0$ (note that since f^2 is strongly holomorphic on Z, we see this as a current on Z, as explained above), while $f^3 \neq gf^2$ for any $g \in \mathcal{O}(Z)$, since $f \notin \mathcal{O}(Z)$. That $f^3\bar\partial(1/f^2) = 0$ can be seen either by going back to the normalization, where we get $t^3\bar\partial(1/t^2)$, which is 0 by the (smooth) duality theorem, or by seeing it as a current in the ambient space, and using the Poincaré-Lelong formula as in Example 1 below.

Let us now consider a germ of a normal variety (Z, z), and the Coleff-Herrera product of one holomorphic function. Assume that $g \in \operatorname{ann}\bar\partial(1/f)$. Since $\bar\partial(1/f)$ is just $\bar\partial$ of $1/f$ in the current sense and g is holomorphic, we get that

$$\bar\partial\left(g\frac{1}{f}\right) = 0.$$

In the smooth case, by regularity of the $\bar\partial$-operator on 0-currents, $g1/f$ would be a holomorphic function. This will not hold in general on a singular space (as the example above shows). However, we get that $g/f \in \mathcal{O}(Z_{\mathrm{reg}})$. If (Z, z) is normal, then $\operatorname{codim}(Z_{\mathrm{sing}}, z) \geq 2$ in Z, and any function holomorphic on an analytic variety outside some subvariety of codimension ≥ 2 is locally bounded, see [10], Proposition II.6.1. Thus, g/f is weakly holomorphic, and since (Z, z) is normal, $g/f \in \mathcal{O}_{Z,z}$, i.e., $g \in \mathcal{J}(f)$. Combined with Theorem 1.2, we get that the duality theorem holds for the Coleff-Herrera product of one holomorphic function on (Z, z) if it is normal.

Assume now that (Z, z) is not normal. Then, there exists $\phi \in \tilde{\mathcal{O}}_{Z,z} \setminus \mathcal{O}_{Z,z}$. Since weakly holomorphic functions are meromorphic, we can write $\phi = g/h$ for some strongly holomorphic functions g and h. Then $g\bar\partial(1/h) = 0$, by Theorem 4.3 in [15] (the analogue of Theorem 1.2 for weakly holomorphic functions). However, since $g/h = \phi \in \tilde{\mathcal{O}}_{Z,z} \setminus \mathcal{O}_{Z,z}$, $g \notin \mathcal{J}(h)$ (in $\mathcal{O}_{Z,z}$).

Hence, in the case of the Coleff-Herrera product of one single holomorphic function on a germ of an analytic variety (Z, z), we get that the duality theorem holds for all f if and only if (Z, z) is normal. The next example shows that this characterization does not extend to tuples of holomorphic functions.

Example 1. Let $Z = \{z_1^2 + \cdots + z_k^2 = 0\} \subseteq \mathbb{C}^k$, where $k \geq 3$. Then Z is normal since Z is a reduced complete intersection with $Z_{\mathrm{sing}} = \{0\}$, and a reduced complete intersection is normal if and only if $\operatorname{codim} Z_{\mathrm{sing}} \geq 2$. Let $\mu = \bar\partial(1/z_{k-1}) \wedge \cdots \wedge \bar\partial(1/z_1)$ (seen as a current on Z). We claim that $z_k\mu = 0$ by considering this as a current in the ambient space, i.e., $i_*(z_k\mu)$, and using the Poincaré-Lelong formula,

$$i_*(z_k\mu) = z_k\bar\partial\frac{1}{z_{k-1}} \wedge \cdots \wedge \bar\partial\frac{1}{z_1} \wedge \frac{1}{2\pi i}\bar\partial\frac{1}{z_1^2 + \cdots + z_k^2} \wedge d(z_1^2 + \cdots + z_k^2).$$

Then, $z_k dz_i^2 = 2z_i z_k dz_i$ and $z_i z_k \in \mathcal{J}(z_1, \cdots, z_{k-1}, z_1^2 + \cdots + z_k^2)$ for $i = 1, \ldots, k$, so each such term annihilates the current by Theorem 1.2. However, $z_k \notin \mathcal{J}(z_1, \cdots, z_{k-1})$ in $\mathcal{O}(Z)$.

We will show that depending on certain singularity subvarieties of the analytic sheaf $\mathcal{O}_Z$, compared to the zero set of f, we can give sufficient (and in many cases necessary) conditions for when the duality theorem holds on an analytic variety. This condition can be seen as a generalization of normality, coinciding with the usual notion of normality in the case $p = 1$.

Given a coherent ideal sheaf $\mathcal{J}$, there exists locally a finite free resolution

$$0 \to \mathcal{O}(E_N) \xrightarrow{f_N} \mathcal{O}(E_{N-1}) \to \ldots \xrightarrow{f_1} \mathcal{O}(E_0) \tag{1.2}$$

of the sheaf $\mathcal{O}/\mathcal{J}$, and this induces a complex of vector bundles

$$0 \to E_N \xrightarrow{f_N} E_{N-1} \to \ldots \xrightarrow{f_1} E_0 \to 0.$$

We define Z_k as the set of points where f_k does not have optimal rank. If $Z = Z(\mathcal{J})$ and $p = \operatorname{codim} Z$, then $Z_1 = \cdots = Z_p = Z$ and $Z_{k+1} \subseteq Z_k$, see [12], Corollary 20.12. If $\mathcal{J} = \mathcal{J}_Z$, the ideal of holomorphic functions vanishing on Z, then we define

$$Z^0 := Z_{\mathrm{sing}} \quad \text{and} \quad Z^k := Z_{p+k} \quad \text{for } k \geq 1, \tag{1.3}$$

where $p = \operatorname{codim} Z$. These sets are in fact independent of the choice of resolution by the uniqueness of minimal free resolutions in a local Noetherian ring, and from Remark 1 in [4], Z^k are independent of the local embedding of Z into $\mathbb{C}^n$. Hence they are intrinsic subvarieties of Z. We will use the convention that $\operatorname{codim} Z^k$ refers to the codimension in Z, while by $\operatorname{codim} Z_k$, we refer to the codimension in the ambient space.

Theorem 1.3. *Let $f = (f_1, \ldots, f_p)$ be a holomorphic mapping on a germ of an analytic variety (Z, z) defining a complete intersection. If* $\operatorname{codim}(Z^k \cap Z_f) \geq k + p + 1$ *for $k \geq 0$, then* $\operatorname{ann} \mu^f = \mathcal{J}(f_1, \ldots, f_p)$.

The proof of Theorem 1.3 is in Section 3.

One might conjecture that this equality of the annihilator and the ideal holds if and only if the conditions in the theorem are satisified. We have not been able to prove this in this generality, but have focused on a slightly weaker formulation of it. To do this, we introduce the notion of p-duality for an analytic variety.

Definition 1. If (Z, z) is a germ of an analytic variety, we say that (Z, z) has *p-duality* if for all $f = (f_1, \ldots, f_p) \in \mathcal{O}_{Z,z}^{\oplus p}$ defining a complete intersection, we have $\operatorname{ann} \mu^f = \mathcal{J}(f_1, \ldots, f_p)$.

Theorem 1.3 implies the following statement:

$$(*) \qquad (Z, z) \text{ has } p\text{-duality if } \operatorname{codim} Z^k \geq p + k + 1, \text{ for } k \geq 0.$$

We believe that the converse of $(*)$ holds, and we will discuss this throughout the rest of this introduction. We show that indeed, in many cases, the converse of $(*)$ holds, and if the condition in $(*)$ is not a precise condition for p-duality, it is at least very close to being so.

We saw above that 1-duality is equivalent to that Z is normal. The condition $\operatorname{codim} Z^k \geq k+2$ in $(*)$ is exactly the condition that Z is normal. This is proved in [17], but can also be seen using the conditions R1 and S2 in Serre's criterion for normality. Indeed, one can verify that the conditions R1 and S2 are equivalent to the condition $\operatorname{codim} Z^k \geq k+2$. Thus, the converse of $(*)$ holds when $p=1$.

Recall that a germ (Z,z) is said to be *Cohen-Macaulay* if the ring $\mathcal{O}/\mathcal{J}_{Z,z}$ is Cohen-Macaulay. More concretely, this means that $\mathcal{O}/\mathcal{J}_{Z,z}$ has a free resolution of length $p = \operatorname{codim}(Z,z)$. Equivalently, $Z^k = \emptyset$ for $k \geq 1$. Hence, if (Z,z) is Cohen-Macaulay, the condition $\operatorname{codim} Z^k \geq p+k$ for $k \geq 0$ becomes just $\operatorname{codim} Z_{\text{sing}} \geq p$. In case (Z,z) is Cohen-Macaulay, the converse of $(*)$ holds.

Proposition 1.4. *Assume that (Z,z) is Cohen-Macaulay and that $\operatorname{codim} Z_{\text{sing}} = k$. If $q \geq k$, then there exists $f = (f_1,\dots,f_q) \in \mathcal{O}_{Z,w}^{\oplus q}$, for some w arbitrarily close to z, defining a complete intersection, and $g \in \mathcal{O}_{Z,w}$ such that $g \in \operatorname{ann} \mu^f$, but $g \notin \mathcal{J}(f_1,\dots,f_q)$.*

Remark 1. In general, we need to move to a nearby germ in order to find the counterexample, however, if Z_{sing} is a complete intersection in Z, we can take $w = z$.

In particular, if (Z,z) is a reduced complete intersection, then (Z,z) is Cohen-Macaulay since the Koszul complex is a free resolution of length $\operatorname{codim}(Z,z)$.

In Example 1, $(Z,0)$ is Cohen-Macaulay (since it is a reduced complete intersection) and $Z_{\text{sing}} = \{0\}$, which has codimension $k-1$ in $(Z,0)$. Proposition 1.4 then says that there exists a complete intersection $f = (f_1,\dots,f_{k-1})$ and $g \notin \mathcal{J}(f_1,\dots,f_{k-1})$ such that $g \in \operatorname{ann} \mu^f$. Then $f = (z_1,\dots,z_{k-1})$ and $g = z_k$ is exactly such an example, while for any complete intersection of codimension $< k-1$, the duality theorem holds by Theorem 1.3.

If (Z,z) is not Cohen-Macaulay, we get the converse of $(*)$ only for the least p such that the condition in $(*)$ is not satisfied.

Proposition 1.5. *Assume that (Z,z) satisfies $\operatorname{codim} Z^k \geq k+p$ for all $k \geq 0$, with equality for some $k \geq 1$. Then there exists $f = (f_1,\dots,f_p) \in \mathcal{O}_{Z,z}^{\oplus p}$ defining a complete intersection, and $g \in \mathcal{O}_{Z,z}$, such that $g \in \operatorname{ann} \mu^f$, but $g \notin \mathcal{J}(f_1,\dots,f_p)$.*

If $p=1$, then the weakly holomorphic functions give rise to counterexamples as described above.

The proofs of Proposition 1.4 and Proposition 1.5 are in Section 6 and Section 7 respectively. To prove Proposition 1.4, we use Theorem 5.3, which says that there exists a tuple ξ of holomorphic $(p,0)$-forms such that

$$[Z] = \sum \xi_i \wedge R_i^Z, \tag{1.4}$$

where $[Z]$ is the integration current on Z, and $R^Z = (R_1^Z, \dots, R_N^Z)$ is a tuple of currents such that $\mathcal{J}_Z = \cap_{i=1}^N \operatorname{ann} R_i^Z$, and the current R^Z is defined by means of a free resolution of $\mathcal{O}/\mathcal{J}_Z$, see Section 2. The existence of such ξ_i is proved in [3], but the tuple ξ is not explicitly given. What we prove in Theorem 5.3 is that if R^Z is the current associated with a *minimal* free resolution, then all ξ_i vanish at Z_{sing}. This result can be seen as one generalization of the Poincaré-Lelong formula from the reduced complete intersection case to the Cohen-Macaulay case. In the reduced complete intersection case, the representation (1.4) is given by the Poincaré-Lelong formula, and since in that case, ξ is explicitly given, the fact that ξ vanish at Z_{sing} follows from the implicit function theorem, see the beginning of Section 5.

Summarizing Theorem 1.3 and Propositions 1.4 and 1.5, we get the following.

Corollary 1.6. *Assume that* $\operatorname{codim} Z^k \geq k+p$ *for all* $k \geq 0$*, with equality for some* k*. Then* (Z,w) *has* q*-duality for* $q < p$ *and all* w *in some neighborhood of* z*, and* (Z,w) *does not have* q*-duality for* $q = p$ *for some* w *arbitrarily close to* z*. In addition, if* $\operatorname{codim} Z_{\text{sing}} = p$*, that is, we have equality for* $k = 0$*, then* (Z,w) *does not have* q*-duality for* $q > p$ *for some* w *arbitrarily close to* z*.*

Proof. The only part that does not follow immediately from Theorem 1.3, Proposition 1.4 and Proposition 1.5 is if $q > p$, (Z,z) is not Cohen-Macaulay but there is equality in $\operatorname{codim} Z^k \geq k+p$ for $k=0$. However, in that case, $\operatorname{codim} Z^0 = p$ and $\operatorname{codim} Z^1 \geq p+1$, so since $Z^0 \supseteq Z^1$, there is some $w \in Z^0$ arbitrarily close to z such that (Z,w) is Cohen-Macaulay (i.e., $w \in Z^0 \setminus Z^1$), and we can apply Proposition 1.4. □

2. Residue currents and free resolutions

We will begin by recalling some facts about residue currents. Let $\mathcal{J}$ be a coherent ideal sheaf, and let (E,f) be a free resolution of the sheaf $\mathcal{O}/\mathcal{J}$ as in (1.2). We will throughout assume that $\mathcal{J}$ has pure dimension, which means that the zero set $Z = Z(\mathcal{J})$ has pure dimension. Mostly, we will use the case when $\mathcal{J} = \mathcal{J}_Z$, the sheaf of holomorphic functions vanishing on the analytic variety Z. In particular, if Z is a reduced complete intersection, and $\mathcal{J}_Z = \mathcal{J}(f_1,\dots,f_p)$, then the Koszul complex of f is a free resolution of $\mathcal{O}/\mathcal{J}_Z$. In [5], Andersson and Wulcan constructed a residue current with annihilator equal to $\mathcal{J}$, a current which will be important in the proofs of the theorems above.

Theorem 2.1. *Let $\mathcal{J}$ be a coherent ideal sheaf of pure dimension with a free resolution (E, f), and let $Z = Z(\mathcal{J})$. If $p = \operatorname{codim} Z$, then, given Hermitian metrics on E, there exists a current $R^E = R^E_p + \cdots + R^E_N$, with $\operatorname{ann} R^E = \mathcal{J}$, where R^E_k are E_k-valued $(0, k)$-currents.*

If Z is an analytic subvariety, we will denote by R^Z the current associated with a free resolution of $\mathcal{J}_Z$ of minimal length. Note that this current is not in general uniquely defined, as it might depend on the choice of metrics.

We will only recall briefly how these currents are defined, see [5] for details. There exists a form u, smooth outside of $Z = Z(\mathcal{J})$ such that if $F \neq 0$ is a holomorphic function vanishing at Z, then R^E is defined by

$$R^E = \bar{\partial}|F|^{2\lambda} \wedge u|_{\lambda=0}, \tag{2.1}$$

where for $\operatorname{Re}\lambda \gg 0$, this is a (current-valued) analytic function in λ, and $|_{\lambda=0}$ denotes the analytic continuation to $\lambda = 0$.

If $f = (f_1, \dots, f_p)$ defines a complete intersection, the Coleff-Herrera product coincides with the so called *Bochner-Martinelli current* of f, as introduced by Passare, Tsikh and Yger in [19] in the smooth case. It was also developed in the case of an analytic variety in [7]. If f defines a complete intersection, the Bochner-Martinelli current of f, denoted R^f, can be defined as the current associated with the Koszul complex of f. In fact, in [5], currents associated with any generically exact complex of vector bundles are defined, and not only free resolutions as in Theorem 2.1, and then the Bochner-Martinelli current for an arbitrary f can be defined as the current associated with the Koszul complex of f, see [1]. This equality of the Coleff-Herrera product and Bochner-Martinelli current makes the Coleff-Herrera product fit in the framework of residue currents associated with a free resolution, and this substitution will be used throughout the arguments. The theorem below is Theorem 4.1 in [19] in the smooth case, and Theorem 6.3 in [15] in the singular case.

Theorem 2.2. *If $f = (f_1, \dots, f_p)$ defines a complete intersection on Z, then the Bochner-Martinelli current R^f of f equals the Coleff-Herrera product μ^f of f.*

Pseudomeromorphic currents were introduced in [6]. A current of the form

$$\frac{1}{z_{i_1}^{k_1}} \cdots \frac{1}{z_{i_m}^{k_m}} \bar{\partial}\frac{1}{z_{i_{m+1}}^{k_{m+1}}} \wedge \cdots \wedge \bar{\partial}\frac{1}{z_{i_p}^{k_p}} \wedge \alpha,$$

where α is a smooth form with compact support, is called an elementary current. A current T is said to be a *pseudomeromorphic current*, denoted $T \in \mathcal{PM}$, if it is a locally finite sum of push-forwards of elementary currents. As can be seen from their construction, the Coleff-Herrera product μ^f and the current R^E associated with a free resolution are pseudomeromorphic. We will need the following property of pseudomeromorphic currents, see Corollary 2.4 in [6].

Proposition 2.3. *If $T \in \mathcal{PM}$ is of bidegree $(0,p)$ and T has support on a variety of codimension $\geq p+1$, then $T = 0$.*

We will use results from [2], that one can define products of the currents R^f and R^Z, and that under certain conditions, the annihilator of the product $R^f \wedge R^Z$ equals the sum of the ideals $\mathcal{J}(f) + \mathcal{J}_Z$. This type of product can be defined more generally for currents R^E and R^F associated with two free resolutions E and F. If R^E is defined by

$$R^E := \bar\partial |G|^{2\lambda} \wedge u|_{\lambda=0},$$

then $R^E \wedge R^F$ can be defined by

$$R^E \wedge R^F := \bar\partial |G|^{2\lambda} \wedge u \wedge R^F|_{\lambda=0}.$$

Remark 2. If we consider $R^f \wedge R^Z$, where $f = (f_1, \dots, f_p)$ is a strongly holomorphic mapping on Z, then this depends a priori on the choice of representatives of f in the ambient space. We will only need that under certain conditions, $\operatorname{ann} R^f \wedge R^Z = \mathcal{J}(f) + \mathcal{J}_Z$, which is independent of the choice of representatives. However, one can in fact show that $R^f \wedge R^Z$ does not depend on the choice of representatives, essentially due to that R^Z is annihilated by both holomorphic and anti-holomorphic functions vanishing on Z.

If

$$0 \to E_n \xrightarrow{f_n} E_{n-1} \to \dots \xrightarrow{f_1} E_0 \to 0$$

and

$$0 \to F_m \xrightarrow{g_m} F_{m-1} \to \dots \xrightarrow{g_1} F_0 \to 0$$

are two complexes, then one can form the tensor product of the complexes, denoted $(E \otimes F, f \otimes g)$, by letting $(E \otimes F)_k = \oplus_{i+j=k} E_i \otimes F_j$ and $(f \otimes g)(\xi \otimes \eta) = f_i \xi \otimes \eta + (-1)^i \xi \otimes g_j \eta$ if $\xi \otimes \eta \in E_i \otimes F_j$.

The following theorem, Theorem 4.1 and Remark 8 in [2], and its corollary gives conditions for when the annihilator of $R^E \wedge R^F$ coincides with the sum of the annihilators, and when the tensor product of two (minimal) free resolutions is a (minimal) free resolution.

Theorem 2.4. *Let (E,f) and (F,g) be free resolutions of ideal sheafs $\mathcal{I}$ and $\mathcal{J}$, and let $Z_k^{\mathcal{I}}$ and $Z_l^{\mathcal{J}}$ be the associated sets where f_k and g_l does not have optimal rank. If* $\operatorname{codim}(Z_k^{\mathcal{I}} \cap Z_l^{\mathcal{J}}) \geq k+l$ *for all $k,l \geq 1$, then* $\operatorname{ann} R^E \wedge R^F = \mathcal{I} + \mathcal{J}$ *and $(E \otimes F, f \otimes g)$ is a free resolution of $\mathcal{I} + \mathcal{J}$. In addition, if both E and F are minimal free resolutions at some point z, then the tensor product is a minimal free resolution.*

To be precise, the last statement is not included in [2]. However, if the tensor product is a free resolution, it follows immediately from the definition of minimality at some z, that $\operatorname{Im} f_k \subseteq \mathfrak{m}_z \mathcal{O}(E_{k-1})$ (where $\mathfrak{m}_z$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,z}$), that it is minimal.

Corollary 2.5. *If $f = (f_1, \dots, f_p)$ is a reduced complete intersection on Z, and* $\operatorname{codim} Z_f \cap Z^l \geq p+l$ *for* $l \geq 1$, *then* $\operatorname{ann} R^f \wedge R^Z = \mathcal{J}(f) + \mathcal{J}_Z$, *and the tensor product of the Koszul complex of f and a free resolution of $\mathcal{J}_Z$ is a free resolution of $\mathcal{J}(f)+\mathcal{J}_Z$. In addition, if the free resolution of $\mathcal{J}_Z$ is minimal at some point z, then the tensor product is a minimal free resolution.*

Proof. If f is a complete intersection, then the Koszul complex of f is a minimal free resolution, and its associated singularity subvarieties Z_k^f are equal to Z_f for $k \leq p$, and empty for $k > p$. Since $Z_l = Z$ for $l \leq \operatorname{codim} Z$, the condition $\operatorname{codim} Z_f \cap Z_l \geq p + l$ is automatic for $l \leq \operatorname{codim} Z$ since f is a complete intersection on Z. Thus, the condition $\operatorname{codim} Z_k^f \cap Z_l \geq k+l$ becomes just $\operatorname{codim} Z_f \cap Z^l \geq p+l$. $\square$

3. Proof of Theorem 1.3

The inclusion $\mathcal{J}(f_1, \dots, f_p) \subseteq \operatorname{ann} \mu^f$ follows from Theorem 1.2 (also without the conditions on $Z^k \cap Z_f$), so we only need to prove the reverse inclusion. Assume that $Z \subseteq \Omega \subseteq \mathbb{C}^n$ and that $\operatorname{codim} Z = q$. Then $i_* \mu^f = \mu^f \wedge [Z]$, where $i : Z \to \Omega$ is the inclusion, and by Theorem 2.2, $\mu^f \wedge [Z] = R^f \wedge [Z]$. We will show that $g \in \operatorname{ann}(R^f \wedge [Z])$ implies that $g \in \operatorname{ann}(R^f \wedge R^Z)$ (which does not hold in general, but does under the conditions of the theorem). By Proposition 2.2 in [4], outside of Z_{sing} there exists a smooth $(q, 0)$-vector field γ such that $\gamma \neg [Z] = R^Z_q$. Then

$$gR^f \wedge R^Z_q = gR^f \wedge (\gamma \neg [Z]) = \gamma \neg (gR^f \wedge [Z]) = 0$$

outside of Z_{sing}. Hence $gR^f \wedge R^Z_q$ is a $(0, p+q)$-current with support on $Z_f \cap Z_{\text{sing}}$, so by Proposition 2.3, it is 0 since $Z_f \cap Z_{\text{sing}}$ has codimension $\geq p + q + 1$.

Outside of Z^{k+1}, there exists a smooth $\operatorname{Hom}(E_{q+k}, E_{q+k+1})$-valued smooth $(0,1)$-form α_{q+k+1} such that $R^Z_{q+k+1} = \alpha_{q+k+1} R^Z_{q+k}$, see [5]. We will prove by induction that

$$gR^f \wedge R^Z_{q+k} = 0. \tag{3.1}$$

Above we proved this for $k = 0$, so let us assume that it is proved for k. Then

$$gR^f \wedge R^Z_{q+k+1} = \alpha_{q+k+1}(gR^f \wedge R^Z_{q+k}) = 0$$

outside of $Z^f \cap Z^{k+1}$. Thus $gR^f \wedge R^Z_{q+k+1}$ has support on $Z^f \cap Z^{k+1}$ which has codimension $\geq p + q + k + 2$, and since it is a pseudomeromorphic current of bidegree $(0, p+q+k+1)$, it is 0 by Proposition 2.3. Thus we have proven that $g \in \operatorname{ann}(R^f \wedge R^Z)$. By Corollary 2.5, $\operatorname{ann}(R^f \wedge R^Z) = \mathcal{J}(f) + \mathcal{J}_Z$, and hence we get that $g \in \mathcal{J}(f) + \mathcal{J}_Z$.

4. Complete intersections and choice of coordinates

This section contains several lemmas about choices of coordinates and existence of complete intersections containing a certain variety. They will be used throughout the rest of the sections. This first lemma, which is based on the first lemma in Section 5.2.2 in [13], is the basis for the rest of them.

Lemma 4.1. *Assume that $(V,z) \subseteq (Z,z)$, where (Z,z) has pure dimension, V has codimension ≥ 1 in Z and that there exists $f = (f_1,\dots,f_m)$ such that $(V,z) = (Z,z) \cap \{f_1 = \cdots = f_m = 0\}$. Then there exists a finite union, E, of proper linear subspaces of $\mathbb{C}^m$, such that $(Z,z) \cap \{a \cdot f = 0\}$ has codimension 1 in (Z,z) if $a \in \mathbb{C}^m \setminus E$.*

Proof. The set E of $a \in \mathbb{C}^m$ such that $(Z,z) \cap \{a \cdot f = 0\} = (Z,z)$ is a linear subspace of $\mathbb{C}^m$, and since $(Z,z) \cap \{f_1 = \cdots = f_m = 0\}$ has positive codimension, it must be a proper subspace. If (Z,z) is irreducible, there thus exists a proper subspace $E \subsetneq \mathbb{C}^m$ such that $(Z,z) \cap \{a \cdot f = 0\}$ has codimension 1 in (Z,z) if $a \in \mathbb{C}^m \setminus E$. If (Z,z) is reducible, then there exists such subspaces E_i for each irreducible component (Z_i,z) of (Z,z), and thus we can take $E = \cup E_i$. □

The following two lemmas are about existence of certain complete intersections containing a given variety, and their existence are the basis for the counterexamples to the duality theorem.

Lemma 4.2. *Assume that $(V,z) \subseteq (Z,z)$, where (Z,z) has pure dimension,* $\operatorname{codim} V = p$ *in Z, and let $f = (f_1,\dots,f_m)$ be such that $(V,z) = (Z,z) \cap \{f_1 = \cdots = f_m = 0\}$. Then there exists $f' = (f'_1,\dots,f'_p)$, a complete intersection on Z, such that $(V,z) \subseteq (V',z) := (Z,z) \cap \{f'_1 = \cdots = f'_p = 0\}$, where $f'_i = \sum a_{i,j} f_j$.*

Proof. By Lemma 4.1, there exists $E \subsetneq \mathbb{C}^m$ such that $(Z,z) \cap \{a \cdot f = 0\}$ has codimension 1 in (Z,z) for $a \in \mathbb{C}^m \setminus E$. We choose $f'_1 = a \cdot f$, for some $a \in \mathbb{C}^m \setminus E$. Proceeding in the same way with $(Z,z) \cap \{f'_1 = 0\}$ instead of (Z,z), we get f'_2 such that $(Z,z) \cap \{f'_1 = f'_2 = 0\}$ has codimension 2 in Z. Repeating this, $f' = (f'_1,\dots,f'_p)$ will be the desired complete intersection. □

Lemma 4.3. *Assume that $(V,z) \subseteq (Z,z)$, where (V,z) has codimension p in (Z,z) and* $\dim(Z,z) = d$. *Then, for some w arbitrarily close to z, there exists a complete intersection $f = (f_1,\dots,f_d) \in \mathcal{O}^{\oplus d}_{Z,w}$ such that $(V,w) = (Z,w) \cap \{f_1 = \cdots = f_p = 0\}$.*

Proof. By Lemma 4.2, there exists $f = (f_1,\dots,f_p)$ a complete intersection on (Z,z) such that $(V,z) \subseteq (V',z)$, where $V' = \{f_1 = \cdots = f_p = 0\}$. Since the set where V' is reducible has codimension $> p$, there exists some w arbitrarily close to z such that $(V,w) = (V',w)$. Then we apply Lemma 4.2 again to $(\{w\},w) \subseteq (V,w)$ to find $(f_{p+1},\dots,f_d)$, a complete intersection on (V, w), so that $f = (f_1, \dots, f_d)$ is the desired complete intersection. □

This last lemma is about the existence of a certain choice of coordinates, which is used in the proof of Theorem 5.3.

Lemma 4.4. *Let $(Z, 0) \subseteq (\mathbb{C}^n, 0)$ and assume that Z has pure dimension d. Then we can choose coordinates w on $\mathbb{C}^n$ such that $(Z, 0) \cap \{w_I = 0\} = \{0\}$ for all $I \subseteq \{1, \dots, n\}$ with $|I| = d$.*

Proof. We will choose the coordinates w on $\mathbb{C}^n$ inductively. By Lemma 4.1, there exists E such that $(Z, 0) \cap \{a \cdot z = 0\}$ has codimension 1 in Z if $a \notin E$, and we choose $w_1 = a \cdot z$ for some $a \notin E$. Now, we assume by induction that we have chosen coordinates $(w_1, \dots, w_k)$ such that $(Z, 0) \cap \{w_I = 0\}$ has codimension $|I|$ for each $I \subseteq \{1, \dots, k\}$ with $|I| \leq d$. For each $I \subseteq \{1, \dots, k\}$ with $|I| \leq d - 1$, we can then find E_I by Lemma 4.1 such that $(Z, 0) \cap \{w_I = 0\} \cap \{a \cdot z = 0\}$ has codimension 1 in $(Z, 0) \cap \{w_I = 0\}$ if $a \notin E_I$. Since each E_I is a finite union of proper subspaces of $\mathbb{C}^n$, we can find $a \in \mathbb{C}^n \setminus \cup E_I$, and we then let $w_{k+1} = a \cdot z$. Proceeding in this way, $w = (w_1, \dots, w_n)$ will be the desired choice of coordinates. □

5. Representations of the integration current in the Cohen-Macaulay case

To prove Proposition 1.4, we will use the following representation of the integration current $[Z]$ on Z in terms of the current R^Z. Assume that Z is Cohen-Macaulay, and that $\operatorname{codim} Z = p$, so that $R^Z = R^Z_p$ by Theorem 2.1. By Example 1, [3], there exist holomorphic $(p, 0)$-forms ξ_i such that

$$[Z] = \sum \xi_i \wedge R^Z_{p,i}, \tag{5.1}$$

where $R^Z_{p,i}$ are the various components of R^Z, i.e., given a local frame $(e_1, \dots, e_N)$ of $\mathcal{O}(E_p)$, $R^Z_p = \sum R^Z_{p,i} e_i$.

If Z is a reduced complete intersection defined by $f = (f_1, \dots, f_p)$, then $R^Z = \mu^f$ by Theorem 2.2, and by the Poincaré-Lelong formula, see [9], we have

$$[Z] = \frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1} \wedge df_1 \wedge \dots \wedge df_p.$$

Thus, we can take $\xi = df_1 \wedge \dots \wedge df_p$, and then it is clear by the implicit function theorem that ξ vanishes at Z_{sing}. We will show that this is the case also when Z is Cohen-Macaulay. This is Theorem 5.3, and the proof will use the following lemmas. Recall that the *socle* of module M over a local ring $(R, \mathfrak{m}, k)$ is defined as $\operatorname{Hom}_R(k, M)$, see [8]. We will use the following characterization of the socle, which is immediate from the definition:

$$\operatorname{Hom}_R(k, M) \cong \{\varphi \in M \mid \mathfrak{m}\varphi = 0\}. \tag{5.2}$$

Lemma 5.1. *Let $\mathfrak{q}$ be a germ of an ideal at 0 such that $\sqrt{\mathfrak{q}} = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal at 0, and let*

$$0 \to \mathcal{O}(E_n) \xrightarrow{f_n} \dots \xrightarrow{f_1} \mathcal{O}(E_0) \to \mathcal{O}/\mathfrak{q} \to 0 \tag{5.3}$$

be a minimal free resolution of $\mathcal{O}/\mathfrak{q}$, where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n,0}$. Then

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}) = \operatorname{rank} E_n.$$

Proof. We have

$$\operatorname{rank} E_n = \dim \operatorname{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$$

since $\operatorname{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ is just the n:th homology of the complex (5.3) tensored with $\mathcal{O}/\mathfrak{m}$. This is $\mathbb{C}^{\operatorname{rank} E_n}$ since the free resolution is minimal so that if

$$\tilde{f}_n : \mathcal{O}(E_n) \otimes \mathcal{O}/\mathfrak{m} \to \mathcal{O}(E_{n-1}) \otimes \mathcal{O}/\mathfrak{m},$$

then $\tilde{f}_n = 0$ since $\operatorname{Im} f_n \subseteq \mathfrak{m}E_{n-1}$ by definition of minimality of a free resolution. However, $\operatorname{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ can also be computed by taking a free resolution of $\mathcal{O}/\mathfrak{m}$, tensoring it with $\mathcal{O}/\mathfrak{q}$ and taking homology. Since the Koszul complex of $(z_1, \dots, z_n)$ is a free resolution of $\mathcal{O}/\mathfrak{m}$, we get

$$\begin{aligned}\operatorname{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}) &\cong \operatorname{Ker}\left(\bigwedge^n \mathcal{O}/\mathfrak{q} \xrightarrow{\delta_z} \bigwedge^{n-1} \mathcal{O}/\mathfrak{q}\right)\\ &\cong \{\varphi \in \mathcal{O}/\mathfrak{q} \mid \mathfrak{m}\varphi = 0\} \cong \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}),\end{aligned}$$

where the last equality is (5.2). □

Lemma 5.2. *Assume that there exists currents $\mu_1, \dots, \mu_N$ such that $\mathfrak{q} = \cap \operatorname{ann} \mu_i$, where $\mathfrak{q}$ is an ideal such that $\sqrt{\mathfrak{q}} = \mathfrak{m}$. Then*

$$N \geq \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}).$$

Proof. We claim that there exists a $\mathbb{C}$-linear injective mapping

$$\tilde{\mu} : \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}) \to \mathbb{C}^N,$$

which proves the statement. We consider $\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ as (5.2). Since $\mathfrak{q} \subseteq \operatorname{ann} \mu_i$, the mapping $\varphi \mapsto \varphi\mu_i, \varphi \in \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ is well-defined. Since $\mathfrak{m}\varphi = 0$, $\varphi\mu_i$ is a current of order 0 with support on $\{0\}$. Thus

$$\varphi\mu_i = a_i R_0, \tag{5.4}$$

for some $a_i \in \mathbb{C}$, where R_0 is the current $\delta_{z=0} d\bar{z}$, that is, $R_0.\varphi dz = \varphi(0)$. We thus get a mapping

$$\tilde{\mu}(\varphi) = (a_1, \dots, a_N),$$

where a_i are defined by (5.4). It only remains to see that $\tilde{\mu}$ is injective. However, if $\tilde{\mu}(\varphi) = 0$, then $\varphi \in \cap \operatorname{ann} \mu_i = \mathfrak{q}$, so $\varphi = 0$ in $\mathcal{O}/\mathfrak{q}$. □

Combining Lemma 5.1 and Lemma 5.2, if f is a complete intersection on Z, where Z is Cohen-Macaulay, then none of the components in the decomposition $R^f \wedge R^Z = \sum R^f \wedge R^Z_{p,i}$ are redundant. This will be a crucial step in the proof of the following theorem.

Theorem 5.3. *Let $Z \subseteq \Omega \subseteq \mathbb{C}^n$ be a subvariety of Ω of codimension p, and assume that Z is Cohen-Macaulay. Then there exists holomorphic $(p,0)$-forms ξ_i such that*

$$[Z] = \sum \xi_i \wedge R^Z_{p,i},$$

and if R^Z is defined with respect to a minimal free resolution of $\mathcal{O}_Z$, then all ξ_i vanish at Z_{sing}.

Proof. As mentioned in the introduction of the section, the existence of ξ_i is Example 1 in [3], so we only need to prove that ξ_i vanish at Z_{sing} if R^Z is defined with respect to a minimal free resolution. Assume that $0 \in Z_{\text{sing}}$. We begin by choosing coordinates in $\mathbb{C}^n$ such that $\{w_J = 0\} \cap Z = \{0\}$ for all $J \subseteq \{1, \dots, n\}$ with $|J| = n - p$, which is possible by Lemma 4.4. We have

$$[Z] = \sum_{i,|I|=p} \xi_{I,i} dw_I \wedge R^Z_{p,i}, \tag{5.5}$$

where $\xi_{I,i}$ are holomorphic functions, and we are done if we can prove that $\xi_{I,i}(0) = 0$ for all $\xi_{I,i}$.

Fix some $I \subseteq \{1, \dots, n\}$ with $|I| = p$. Let $w' = (w_{J_1}, \dots, w_{J_{n-p}})$, where $J = I^c$. By the Poincaré-Lelong formula applied to w' on Z, see [9], Section 1.9, we have that

$$\frac{1}{(2\pi i)^p} R^{w'} \wedge dw' \wedge [Z] = k[0]$$

for some $k \geq 1$. Combined with the Poincaré-Lelong formula applied to w in $\mathbb{C}^n$, we get

$$R^w \wedge dw = (2\pi i)^n [0] = ((2\pi i)^{n-p}/k) R^{w'} \wedge dw' \wedge [Z].$$

Since by (5.5)

$$dw' \wedge [Z] = \pm \sum_i \xi_{I,i} dw \wedge R^Z_{p,i}$$

we get that

$$R^w = C \sum_i \xi_{I,i} R^{w'} \wedge R^Z_{p,i} \tag{5.6}$$

for some constant $C \neq 0$.

We first consider the case when R^Z consists of one single component R^Z_p. By Corollary 2.5, $\operatorname{ann}(R^{w'} \wedge R^Z_p) = \mathcal{J}(w') + \mathcal{J}_Z$. We claim that the inclusion $\mathcal{J}(w)_0 \supseteq (\mathcal{J}(w') + \mathcal{J}_Z)_0$ is strict. If the inclusion is not strict, then w' generates the maximal ideal $\mathfrak{m}_{Z,0}$ in $\mathcal{O}_{Z,0}$, which is a contradiction by Proposition 4.32 in [10], since the number of functions needed to generate the maximal ideal at a singular point must be strictly larger than the dimension. Thus there exists a g in

$$\mathcal{J}(w)_0 \setminus (\mathcal{J}(w') + \mathcal{J}_Z)_0 = (\operatorname{ann} R^w)_0 \setminus (\operatorname{ann}(R^{w'} \wedge R^Z_p))_0.$$

Multiplying (5.6) by g, we get that $g\xi_I \in \operatorname{ann}(R^{w'} \wedge R^Z_p)$, and hence we must have $\xi_I(0) = 0$.

Now we consider the case when R^Z_p consists of more than one component. By Corollary 2.5, the tensor product of the Koszul complex of w' and the minimal free resolution of $\mathcal{J}_Z$ is a minimal free resolution of $\mathfrak{q} := \mathcal{J}(w') + \mathcal{J}_Z$, and the rank N of its left-most non-zero module is equal to the rank of the left-most non-zero module in the free resolution of $\mathcal{J}_Z$ since the left-most non-zero module of the Koszul complex has rank 1. By Corollary 2.5, we have

$$\mathfrak{q} = \cap_{i=1}^N \operatorname{ann}(R^{w'} \wedge R^Z_{p,i}). \tag{5.7}$$

By Lemma 5.1, $N = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ and by Lemma 5.2, if $\mathfrak{q} = \cap_{i=1}^m \operatorname{ann} \mu_i$, then $m \geq N$. Thus, if we remove one term $\operatorname{ann}(R^{w'} \wedge R^Z_{p,i})$ from the intersection in (5.7), we get something strictly larger, i.e., for any i,

$$(\cap_{j\neq i} \operatorname{ann}(R^{w'} \wedge R^Z_{p,j})) \setminus (\operatorname{ann} R^{w'} \wedge R^Z_{p,i}) \neq \emptyset. \tag{5.8}$$

We fix some $i = 1, \dots, n$, and take g_i in (5.8) and multiply (5.6) by g_i. Since $g_i \in \cap_{j\neq i} \operatorname{ann}(R^{w'} \wedge R^Z_{p,j})$, we must have $g_i \in \mathfrak{m}$, so $g_i R^w = 0$. Thus we get

$$g_i \xi_{I,i} R^{w'} \wedge R^Z_{p,i} = 0.$$

Since $g_i \notin \operatorname{ann}(R^{w'} \wedge R^Z_{p,i})$ but $g_i\xi_{I,i} \in \operatorname{ann}(R^{w'} \wedge R^Z_{p,i})$, we must have $\xi_{I,i} \in \mathfrak{m}$, and we are done. $\square$

6. Proof of Proposition 1.4

By moving to a nearby germ (Z, w), we can assume that Z_{sing} has pure codimension k, and that there exists a complete intersection $f = (f_1, \dots, f_q)$ on (Z, w) such that $(Z_{\text{sing}}, w) = \{f_1 = \cdots = f_k = 0\} \cap (Z, w)$, see Lemma 4.3. We let $\mathcal{I} = \mathcal{J}(f_1, \dots, f_q)_w$ and $V = Z(\mathcal{I})$, and since $q \geq k$, $V \subseteq Z_{\text{sing}}$. Since $\mathcal{J}_{V,w}$ is finitely generated over $\mathcal{O}_{Z,w}$, we get from the Nullstellensatz that $\mathcal{J}^m_{V,w} \subseteq \mathcal{I}$ for m sufficiently large. Now, we choose m to be minimal such that this inclusion holds. Thus, there exists a function $g \in \mathcal{J}^{m-1}_{V,w} \setminus \mathcal{I}$, such that $g\mathcal{J}_{V,w} \subseteq \mathcal{I}$. Since $g \notin \mathcal{I}$, we are done if we can show that $g\mu^f \wedge [Z] = 0$.

By Theorem 2.2, we can replace μ^f by R^f, and instead show that $gR^f \wedge [Z] = 0$. By (5.1) and Theorem 5.3

$$gR^f \wedge [Z] = g \sum \xi_i \wedge R^f \wedge R^p_i,$$

where ξ_i are holomorphic $(p, 0)$-forms vanishing on Z_{sing}. Thus $\xi_i = \sum \xi_{I,i} dw_I$, where $\xi_{I,i}$ are holomorphic functions vanishing at Z_{sing}.

Since $g\mathcal{J}_{V,w} \subseteq \mathcal{I}$ and $\mathcal{J}_{Z_{\rm sing},w} \subseteq \mathcal{J}_{V,w}$, we get that $g\xi_{I,i} \in \mathcal{I}$ in $\mathcal{O}_{Z,w}$. By Corollary 2.5, $\operatorname{ann} R^f \wedge R^Z = \mathcal{I} + \mathcal{J}_{Z,w}$. Since if $g\xi_{I,i} \in \mathcal{I}$ in $\mathcal{O}_{Z,w}$, then $g\xi_{I,i} \in \mathcal{I} + \mathcal{J}_{Z,w}$ in $\mathcal{O}_{\mathbb{C}^n,w}$, we get that $gR^f \wedge [Z] = 0$.

7. Singularity subvarieties and counterexamples in the non Cohen-Macaulay case

We will recall the notion of singularity subvarieties of analytic sheafs from [20]. Let R be a local Noetherian ring and $M \neq 0$ a finitely generated R-module. A *regular M-sequence* in an ideal $I \subseteq R$ is a sequence $(f_1, \dots, f_p)$ in I such that f_i is not a zero-divisor in $M/(f_1, \dots, f_{i-1})M$ for $i = 1, \dots, p$. The *depth* of an ideal I on a module M, denoted $\operatorname{depth}_I M$ is the maximal length of a regular M-sequence in I. By $\operatorname{depth}_R M$, we will denote the depth of the maximal ideal $\mathfrak{m}$ of R on M. This is also called the homological codimension of R. The *homological dimension* of M, denoted $\operatorname{dh}_R M$, is defined as the minimal length of any free resolution of M.

A *regular local ring* is a local ring R such that the maximal ideal $\mathfrak{m}$ of R is generated by $n = \dim R$ elements, where $\dim R$ is the Krull-dimension of R, that is, the maximal length of a strict chain of prime ideals in R. In particular, if Z is an analytic variety, then $\mathcal{O}_{Z,z}$ is a regular local ring if and only if $z \in Z_{\rm reg}$, see Proposition 4.32 in [10]. The following is Theorem 19.9 in [12].

Proposition 7.1. *If R is a regular local ring, and M is a finitely generated R-module, then* $\operatorname{dh}_R M + \operatorname{depth}_R M = \dim R$.

Let $\mathcal{F}$ be a coherent analytic sheaf on $\Omega \subseteq \mathbb{C}^n$, and let $\mathcal{O}_z$ denote the ring of germs of holomorphic functions at z in Ω. The *singularity subvarieties*, S_m, of $\mathcal{F}$ are defined by

$$S_m(\mathcal{F}) = \{z \in \Omega; \operatorname{depth}_{\mathcal{O}_z} \mathcal{F}_z \leq m\},$$

where we use the convention that $\operatorname{depth}_R M = \infty$ if $M = 0$, so that $S_m \subseteq \operatorname{supp} \mathcal{F}$. We will use the following alternative definition of the sets Z_k associated with an analytic sheaf above:

$$Z_k(\mathcal{F}) = \{z \in \Omega; \operatorname{dh}_{\mathcal{O}_z} \mathcal{F}_z \geq k\} \tag{7.1}$$

(in the introduction, we defined the sets Z_k if $\mathcal{F}$ was of the form $\mathcal{O}/\mathcal{J}$, where $\mathcal{J}$ was an coherent ideal sheaf, but the same definition works for any coherent analytic sheaf). To see this, note first that if $\operatorname{rank} F_k(z)$ is constant in a neighborhood of some $z_0 \in \Omega$, then $\mathcal{O}(E_{k-1})/\operatorname{Im} F_k$ is free in a neighborhood of z_0, and conversely, by the uniqueness of minimal free resolutions, $\operatorname{rank} F_k$ must be constant in a neighborhood of z if $k > \operatorname{dh}_{\mathcal{O}_z} \mathcal{F}_z$

Proposition 7.2. *If $\mathcal{F}$ is coherent analytic sheaf on some open set in $\mathbb{C}^n$, we have $S_k(\mathcal{F}) = Z_{n-k}(\mathcal{F})$.*

Proof. This follows from Proposition 7.1 and (7.1). □

Let $\Omega \subseteq \mathbb{C}^n$ be an open set, A a subvariety of Ω with ideal sheaf $\mathcal{J}_A$, and $\mathcal{F}$ a coherent analytic sheaf in Ω. For $z \in \Omega$, we define

$$\operatorname{depth}_{A,z} \mathcal{F} = \begin{cases} \infty & \text{if } \mathcal{F}_z = 0 \\ \operatorname{depth}_{\mathcal{J}_{A,z}} \mathcal{F} & \text{otherwise} \end{cases}.$$

and

$$\operatorname{depth}_A \mathcal{F} = \inf_{z \in A} \operatorname{depth}_{A,z} \mathcal{F}$$

The following is (part of) Theorem 1.14 in [20].

Theorem 7.3. *Let $\Omega \subseteq \mathbb{C}^n$ be some open set, A a subvariety of Ω, and $\mathcal{F}$ a coherent analytic sheaf in Ω. Then for $q \geq 1$, we have* $\operatorname{depth}_A \mathcal{F} \geq q$ *if and only if* $\dim A \cap S_{k+q}(\mathcal{F}) \leq k$ *for all k.*

In particular, if we let Z be an analytic subvariety of Ω, $\mathcal{F} = \mathcal{O}_Z$, and $A = Z^1$, where the sets Z^k associated with Z are defined as in (1.3), we get the following.

Corollary 7.4. *For $q \geq 1$, we have* $\operatorname{depth}_{Z^1} \mathcal{O}_Z \geq q$ *if and only if* $\operatorname{codim} Z^k \geq q + k$ *in Z for all $k \geq 1$*

Proof. If we apply Theorem 7.3 to $A = Z^1$ and $\mathcal{F} = \mathcal{O}_Z$, then we only need to prove that $\operatorname{codim} Z^k \geq q + k$ for $k \geq 1$ is equivalent to $\dim Z^1 \cap S_{k+q}(\mathcal{O}_Z) \leq k$. We can write the last condition as $\dim(Z^1 \cap Z_{n-k-q}) \leq k$ by Proposition 7.2. If we replace $\dim V$ by $n - \operatorname{codim} V$ and set $k' = n-k-q$, we get $\operatorname{codim}(Z^1 \cap Z_{k'}) \geq q+k'$. Since $Z_k = Z$ for $k \leq p$, where $p = \operatorname{codim} Z$, and $Z^1 = Z_{p+1}$, this condition for $k \leq p$ is equivalent to $\operatorname{codim} Z_{p+1} \geq p+q+1$ (in Ω), and since $Z_k \subseteq Z_{p+1} = Z^1$ for $k > p+1$, this is equivalent to $\operatorname{codim} Z_{p+k} \geq p+q+k$ for $k \geq 2$. □

In $\mathbb{C}^n$, it is a standard result that a tuple $f = (f_1, \dots, f_p)$ of holomorphic functions is a complete intersection if and only if it is a regular sequence. However, Corollary 7.4 says that this is not always the case on a singular variety. We will illustrate this with an example.

Example 2. Let $\pi(t_1, t_2) = (t_1, t_1 t_2, t_2^2, t_2^3)$, and let $Z = \pi(\mathbb{C}^2)$. Then $Z_{\text{sing}} = \{0\}$, because outside of $\{t_1 = t_2 = 0\}$, one can construct a holomorphic inverse to π, and we will see that Z is not normal at 0, so $0 \in Z_{\text{sing}}$. The function f such that $\pi^* f = t_2$ is weakly holomorphic on Z, since when $t_1 \neq 0$, $f = z_2/z_1$, and when $t_2 \neq 0$, $f = z_4/z_3$, so that $f \in \mathcal{O}(Z_{\text{reg}})$, and it is clear that f is locally bounded near $Z_{\text{sing}} = \{0\}$. However, f is not strongly holomorphic at 0, because if $f = h$ on Z in a neighborhood of 0, where h is holomorphic in a neighborhood of 0 in $\mathbb{C}^4$, then by taking pull-back by π to $\mathbb{C}^2$, we get

$$t_2 = h(t_1, t_1 t_2, t_2^2, t_2^3),$$

which can be seen to be impossible by a Taylor expansion of h at 0.

Since Z has pure dimension, $\operatorname{codim} Z^k \geq k+1$ for $k \geq 1$ by [12], Corollary 20.14b. Hence, $Z^k = \emptyset$ for $k \geq 2$. Since Z is not normal, it does not satisfy the condition

$$\operatorname{codim} Z^k \geq k+2, \quad k \geq 0 \tag{7.2}$$

for normality (see the introduction). However, since $Z^0 = Z_{\text{sing}} = \{0\}$, the condition (7.2) is satisfied for all $k \neq 1$. Thus, since $Z^1 \subseteq Z_{\text{sing}}$, and $\operatorname{codim} Z^1 \not\geq 3$, we must have $Z^1 = \{0\}$. By Corollary 7.4, there does not exist a regular $\mathcal{O}_Z$-sequence $f = (f_1, f_2)$ in $\mathcal{J}_{Z^1}$, since any such sequence has length ≤ 1. In particular, if we take $f = (z_1, z_3)$, then f is a complete intersection since $Z \cap \{z_1 = z_3 = 0\} = \{0\}$, but f is not a regular sequence. We claim that one can also see this more directly. To begin with, it is clear that $z_3 \notin (z_1)$ in $\mathcal{O}_Z$ since $Z \cap \{z_1 = 0\} \not\subseteq Z \cap \{z_3 = 0\}$. We also have that $z_2 \notin (z_1)$ in $\mathcal{O}_Z$, since if $z_2 \in (z_1)$, then by taking pull-back to $\mathbb{C}^2$ as above, we get

$$t_1 t_2 = t_1 h(t_1, t_1 t_2, t_2^2, t_2^3),$$

which is easily seen to be impossible. However, since $z_2 z_3 = z_1 z_4$ in $\mathcal{O}_Z$, we get that $z_2 z_3 \in (z_1)$ in $\mathcal{O}_Z$. Thus, z_3 is a zero-divisor in $\mathcal{O}_Z/(z_1)$, i.e., (z_1, z_3) is not a regular $\mathcal{O}_Z$-sequence in $\mathcal{J}_{Z^1}$.

Lemma 7.5. *Let $f = (f_1, \dots, f_k)$ be a complete intersection on (Z, z). If*

$$\operatorname{ann}\left(\bar\partial \frac{1}{f_r} \wedge \dots \wedge \bar\partial \frac{1}{f_1}\right) = \mathcal{J}(f_1, \dots, f_r) \text{ for all } r < k,$$

then $(f_1, \dots, f_k)$ is a regular $\mathcal{O}_{Z,z}$-sequence.

Proof. If $k = 1$, this is clear since $\mathcal{O}_{Z,z}$ is reduced and f is assumed to be a complete intersection. By induction over k, we can assume that $(f_1, \dots, f_{k-1})$ is a regular $\mathcal{O}_{Z,z}$-sequence. Assume that $(f_1, \dots, f_k)$ is not a regular sequence in $\mathcal{O}_{Z,z}$. Then, since $f_k \notin \mathcal{J}(f_1, \dots, f_{k-1})$, there exist $g \notin \mathcal{J}(f_1, \dots, f_{k-1})$ such that $f_k g \in \mathcal{J}(f_1, \dots, f_{k-1})$. But since $g \in \mathcal{J}(f_1, \dots, f_{k-1})$ outside of $\{f_k = 0\}$, we get that

$$\operatorname{supp}\left(g \bar\partial \frac{1}{f_{k-1}} \wedge \dots \wedge \bar\partial \frac{1}{f_1}\right) \subseteq \{f_1 = \dots = f_k = 0\}$$

by Theorem 1.2. But then by Proposition 2.3, we get that

$$g \in \operatorname{ann} \bar\partial \frac{1}{f_{k-1}} \wedge \dots \wedge \bar\partial \frac{1}{f_1} = \mathcal{J}(f_1, \dots, f_{k-1}),$$

which is a contradiction. $\square$

Proof of Proposition 1.5. By Lemma 4.2, there exists a complete intersection $(f_1, \dots, f_{p+1})$ such that $Z^1 \subseteq \{f_1 = \dots = f_{p+1} = 0\}$. By Corollary 7.4, $(f_1, \dots, f_{p+1})$ is not a regular $\mathcal{O}_{Z,z}$-sequence in $\mathcal{J}(f_1, \dots, f_{p+1})_z$. Thus by Lemma 7.5, we must have that

$$\operatorname{ann}\left(\bar\partial \frac{1}{f_k} \wedge \dots \wedge \bar\partial \frac{1}{f_1}\right) \supsetneq \mathcal{J}(f_1, \dots, f_k) \tag{7.3}$$

for some $k \leq p$. However, by Theorem 1.3, we have equality for $k \leq p-1$. Thus we must have strict inclusion in (7.3) for $k = p$. $\square$

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